INDEPENDENT SETS AND TREE STRUCTURE

By

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The relationship between a bivariegated graph and a graph with a 1-factor is discussed. Also discussed are the relationship between the structure of a tree and its maximal independent sets, and a method for counting the total number of maximal independent sets for certain trees, including those trees with 1-factors having twice as many points as endpoints.

The greatest number of maximal independent sets a tree on n vertices can have is $2^{[\frac{n}{2}]}$ if n is odd, and $2^{\frac{n}{2}-1}$ + 1 if n is even. Examples are given to show that those bounds are the best possible.

A. R. Bednarek
Chairman

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A <u>graph</u> G is a finite nonempty set V of points together with
a set E of unordered pairs of distinct points of V, called edges. A
tree is a connected graph with no cycles. An independent set of a
graph (or tree) is a set of vertices such that no two are joined by
an edge. Consider the following question: What is the relationship
between the structure of a tree T and its independent sets? To be
more specific, is the structure of T completely determined by the
sequence $(\xi_1, \xi_2, \ldots, \xi_{n-1}; \beta)$ where ξ_i is the number of maximal independent
sets with i vertices, where β is the number of endpoints of T, and
$|T| = n$? This paper studies the latter question, answering it in the
negative, but finding other information about the structure of trees
and its relationship to independent sets.

Chapter I presents some basic definitions and elementary results
of graph theory that are useful in later chapters. The definitions
primarily follow the notation used by Harary [4].

Chapter II introduces the ideas of bivariegation and 1-factor,
and discusses their relationship in the light of a previous result [2]
which characterizes bivariegated trees in terms of the size of the
largest maximal independent sets of the trees. (A tree is bivariegated
if the vertex set can be partitioned into two equal sets such that each
vertex is adjacent to one and only one vertex in the set not containing
it. A tree has a 1-factor if there is a spanning subtree in which all
vertices have degree 1.)

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Chapter III indicates how maximal independent sets can be counted in certain trees; relates number of maximal independent sets to structure; and proves a number theoretic equation using a graph theoretic argument.

Chapter IV proves the fact that in a tree with n vertices, there are at most $2^{[\frac{n}{2}]}$ maximal independent sets, in answer to a question posed by Erdös.

The final chapter discusses the results, questions arising from them, and difficulties involved.

In the appendices are a listing of trees with 1-factors on 2, 4, 6, 8, 10 and 12 vertices; examples on 16, 18, and 20 vertices showing that the sequence described above does not characterize trees completely, and a table comparing $\sigma(T)$ and $\sigma(p(T))$ (see Chapter III) for $|T| \leq 10$.

BASIC DEFINITIONS AND THEORY

Definition 1.1: A graph G is a nonempty finite set of points, V, along with a prescribed set E of unordered pairs of distinct points of V, known as edges. We write G = (V,E).

If two distinct points, x and y, of a graph are joined by an edge, they are said to be adjacent, and we write x adj y.

A walk of a graph G is a finite sequence of points such that each point of the walk is adjacent to the point of the walk immediately preceding it and to the point immediately following it. If the first and last points of a walk are the same point, we say the walk is closed, or is a cycle (provided there are three or more distinct points; and all points are distinct except the initial and final points). A graph is acyclic if it contains no cycles. A walk is a path if all the points are distinct.

A graph is connected if every pair of points is joined by a path.

A tree is a connected, acyclic graph.

Any acyclic graph is a forest, the components of which are trees.

Theorem 1.2: The following are equivalent:

    (1) T is a tree.

    (2) Every two points of T are connected by a unique path.

    (3) T is connected and $p = q + 1$, where $|V| = p$ and $|E| = q$.

(4)   T is acyclic and p = q + 1, where p is the number of points in the vertex set, V, of T, and q is the number of edges.

The proof of Theorem 1.2, as well as the proofs of Theorems 1.4 and 1.6 and of Lemma 1.7 can be found in [9].

Definition 1.3: The degree of a point v of G, denoted deg v, is the number of lines incident with v.

Theorem 1.4: The sum of the degrees of the points, v_i, of a graph G is twice the number of lines, q; that is,

$$\sum_{v_i \in G} \deg v_i = 2q.$$

Definition 1.5: An endpoint of a tree is a point whose degree is one.

Theorem 1.6: Every tree has at least two endpoints.

Notation: T - {x} means T minus the vertex x and all its incident edges.

Lemma 1.7: If T is a tree and if e is an endpoint of T, T - {e} is still a tree.

Remark 1.8: By definition of T - {x}, and the extension of that definition to the removal of any finite number of vertices and their incident edges (T - $\{x_1, \ldots, x_n\}$), it is clear that removing n vertices and their incident edges on vertex at a time is the same as removing the vertices and edges all at one time; i.e., T - $\{x_1, x_2\}$ = (T-$\{x_1\}$) - $\{x_2\}$, and inductively, T - $\{x_1, x_2, \ldots, x_n\}$ = $\underbrace{(\ldots ((\text{T-}\{x_1\}) - \{x_2\}) \ldots -}_{\text{n parentheses}}$

$\{x_n\}$).

Definition 1.9: A subgraph of a graph G is a graph having all of its points and lines in G. A spanning subgraph is a subgraph which contains all the points of G, but not necessarily all the lines.

Definition 1.10: The distance $d(x,y)$ between any two points x and y in a graph G is the length (number of edges) of the shortest path joining them, if any; if not, $d(x,y) = \infty$. A shortest path between x and y is called a geodesic. The length of a longest geodesic in a connected graph G is called the diameter of G, $d(G)$. A diametral path is a path whose length is the diameter of G.

Definition 1.11: Let e be an endpoint of tree T, and let x be the unique point adjacent to e. If deg x = 2, then e is called a remote end vertex.
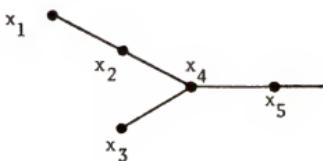

Figure 1

In figure 1, x_1 is a remote end vertex, while x_3 is not since deg x_4 = 3 > 2.

Definition 1.12: An independent set for graph G is a set of vertices with the property that no two vertices in the set are adjacent.

A maximal independent set of G (MIS) is an independent set of G which is contained in no other independent set of G.

A largest maximal independent set (LMIS) of G is a maximal independent set of G with the largest cardinality.
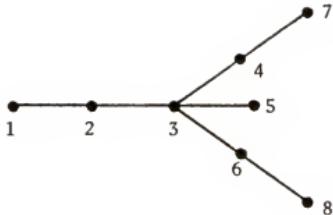
Figure 2

In figure 2, the set {1,5,8} is an independent set. {1,3,7,8} is a MIS which is also a largest maximal independent set, as is {2,4,5,6}.

Lemma 1.13: There is a LMIS of tree T which contains all the endpoints of T.

Proof: Let M be a LMIS of T not containing all the endpoints. Let x be any member of M which is not an endpoint. The point x can be adjacent to at most one endpoint. If x is adjacent to e_1 and e_2, endpoints, then since deg e_1 = 1, and deg e_2 = 1, e_1 and e_2 are not connected to any other point of M. Thus M - {x}∪{e_1}∪{e_2} is an independent set with one more point than M, contradicting our definition of M as a largest independent set. Therefore, no member of M is adjacent to more than one endpoint of T. On the other hand, every endpoint of T not in M is adjacent to a member of M. If not, M∪{e} is independent, again contradicting our definition of M. So let x be the point in M adjacent to e, e ∉ M. If we replace x by e in M, we still have an independent set with the same cardinality as M. Therefore M - {x}∪{e} is also a LMIS, since M is a LMIS and M and M - {x}∪{e} have the same cardinality.

Since there are only finitely many endpoints and hence only a finite number of endpoints not in M, after a finite number of steps like that above, we will have constructed a LMIS containing all the endpoints of T.

Let L_T be the largest maximal independent set of tree T, such that L_T contains all the endpoints of T. Let $\overline{L_T}$ denote the set theoretic complement of L_T in T.

<u>Lemma 1.14</u>: If a tree T has a remote end vertex e with adjacent point x, and $|L_T| = k$, then $|L_{T-\{e,x\}}| = k-1$.

<u>Proof</u>: Certainly $L_T - \{e,x\} \subset T - \{e,x\}$ and $L_T - \{e,x\}$ is an independent set, so that $|L_{T-\{e,x\}}| \geq k-1$. ($|L_T-\{e,x\}| = k-1$ since $e \epsilon L_T$, hence $x \epsilon \overline{L_T}$.) Suppose there is an independent set in $T - \{e,x\}$ of size k. Then e will be independent of all those points, causing $|L_T| = k+1$, a contradiction. Therefore, $|L_{T-\{e,x\}}| = k-1$.

<u>Definition 1.15</u>: A <u>complete graph</u> K_n is a graph on n points such that every pair of points is adjacent. Thus K_n has $\binom{n}{2}$ lines and each point has degree n-1.

<u>Definition 1.16</u>: Let G be a graph with n points. The <u>complement</u> of G is a graph $\overline{G}$ with the same set of vertices as G but whose edge set is the complement in K_n of the edge set of G. See figure 3 for an example.

G: $\overline{G}$:

Figure 3

CHAPTER II

ON 1-FACTORS

Definition 2.1: A graph G = (V,E) is said to be bivariegated if
$G = G_1 - f - G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E_f)$ where $G_1 = (V_1, E_1)$,
$G_2 = (V_2, E_2)$, $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, $f: V_1 \rightarrow V_2$ is a bijection,
and $E_f = \{(x, f(x)) | x \in V_1\}$ where we let $(x, f(x))$ denote the edge
incident with x and f(x). Therefore, a graph G (or tree T) is
bivariegated if and only if its vertex set can be partitioned into
two disjoint equal sets such that each vertex is adjacent to one and
only one vertex in the set not containing it.

The following are properties of a bivariegated tree T:

(1) T has an even number of points and no two endpoints of T
are adjacent to the same points.

(2) T contains at least two remote end vertices.

(3) If e is a remote end vertex of T and e adj x, then T - {e,x}
is also bivariegated. In fact, we can construct any bivariegated tree
by starting with the smallest one, ●—●, and repeatedly adding remote
end vertices. (There is a list of all bivariegated trees on 2, 4, 6,
8, 10, and 12 vertices in Appendix II.)

(4) A bivariegation of T is unique; that is, the partition of
V is unique.

There is a characterization of bivariegated trees in terms of its
maximal independent sets:

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Theorem 2.2: Let T be a tree with $|T| = 2n$, n a natural number. Then T is bivariegated if and only if the largest maximal independent set of T has n elements.

This is proved in [2].

Definition 2.3: An n-factor of a graph G is a spanning subgraph of G which is not totally disconnected and is regular of degree n (that is, every vertex has degree n with respect to the spanning subgraph). In particular, G has a 1-factor if it has a spanning subgraph consisting of disjoint edges.

Tutte [10] characterized graphs with 1-factors:

Theorem 2.4: A graph G has a 1-factor if and only if the number of points of G is even and there is no set S of points such that the number of odd components of G - S exceeds $|S|$.

For trees, we also have

Theorem 2.5: A tree T is bivariegated if and only if T has a 1-factor.

Proof: Suppose T is a bivariegated tree. Then $|T| = 2n$ and T = $V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, such that each vertex in V_1 is adjacent to precisely one vertex in V_2. The edges that define this bijection between V_1 and V_2 are the edges which compose a 1-factor of T. Since the map is a bijection, the edges will be pairwise disjoint; and every point of T is incident with one and only one of these edges.

Conversely, if T has a 1-factor, then $|T| = 2n$ for some positive integer n. Let the set of edges of the 1-factor be the set F. Each point of T is incident with one and only one edge of F. Partition the vertices of T as follows: for some edge $f = (v_{f_1}, v_{f_2})$ in F, put v_{f_1}

in V_1 and v_{f_2} in V_2. Next, place all other vertices adjacent to v_{f_1} in V_1 (none of these will be adjacent to v_{f_2} since T is acyclic). Note that no other edge of F is incident to v_{f_1}. Since each of the vertices adjacent to v_{f_1} are incident with a unique edge of F, place the <u>other</u> vertices incident with these edges in V_2. Continue the partitioning by putting points adjacent to points of V_1 in V_1 and the points adjacent to these new points via edges of F into V_2. This will partition T into V_1 and V_2, $V_1 \cap V_2 = \emptyset$. Every point in V_1 will be adjacent to one and only one point in V_2, since T is acyclic. Thus, T is bivariegated, with the edges of F being the edges of the bijection between V_1 and V_2.

We now have the characterization of trees with 1-factors:

<u>Theorem 2.6</u>: A tree T, $|T| = 2n$, has a 1-factor if and only if the largest maximal independent set of vertices of T contains n vertices.

Note that having a 1-factor and being bivariegated are not equivalent for an arbitrary graph G. G bivariegated implies that G has a 1-factor, since the edges that define the bijection between the two vertex sets will be the lines of the 1-factor. However, the converse is not true. In figure 4, G has a 1-factor but is not bivariegated.

Figure 4

<u>Definition 2.7</u>: A graph G is said to have a <u>spanning subtree</u> if there is a connected acyclic subgraph of G whose vertex set is the vertex set of G.

<u>Corollary to Theorem 2.5</u>: A connected graph G has a 1-factor if and only if G has a bivariegated spanning subtree.

<u>Proof</u>: If G has a bivariegated spanning subtree, then it has a spanning subtree with a 1-factor. Since all points of G are contained in the subtree, then the 1-factor of the tree is also the 1-factor of G.

Now let F be the set of edges of the 1-factor of G. Add edges from G to complete a (connected) tree. It will be a spanning tree since all the points of G are incident with F and it has a 1-factor and is therefore bivariegated. And such a tree can be found since G is connected.

If G has a bivariegated spanning tree and $|G| > 4$, there exists a path α of length 4 or greater, since any bivariegated tree on more than 4 vertices has a diameter of 4 at least.

<u>Definition 2.8</u>: A <u>Hamiltonian path</u> is a path of G containing every vertex of G.

Since any path with an even number of vertices is a bivariegated tree, then any graph with an even number of vertices and a Hamiltonian path has a bivariegated spanning tree.

Ore [7] states the following criterion for a graph having Hamiltonian path:

<u>Theorem 2.9</u>: If a graph G with n vertices deg v_i + deg $v_j \geq n - 1$ for any pair of vertices then G has a Hamiltonian path.

In addition to the partition of the vertex set of a bivariegated tree into two sets according to the bijection, there is a partition into two largest maximal independent sets:

Theorem 2.10: If T is a tree with 2n points which has a 1-factor, then there exist two disjoint unique largest maximal independent sets (LMIS's).

Proof: The statement is certainly true for n = 1 since the only tree on 2 vertices with a 1-factor is ●—●. Assume it holds for all trees with 1-factors on fewer than 2n vertices and let T be a tree with 2n points such that T has a 1-factor. By property (2) of bivariegated trees (and hence of trees with 1-factors), T has a remote end vertex e, with e adj x and x adj y, $y \neq e$. T - {e,x} is a tree with 2n - 2 = 2(n-1) vertices. Since e is an end vertex, then the edge {e,x} must be one of the edges of the 1-factor of T, so that the 1-factor of T minus the edge {e,x} is the 1-factor of T - {e,x}. By the inductive hypothesis, T - {e,x} has 2 disjoint unique largest maximal independent sets, I_1 and I_2, both of size n - 1. One of these sets must contain y, say $y \in I_1$. Then if we return points e and x and the corresponding edges to the tree, the disjoint independent sets of T will be $I_1 \cup \{e\}$ and $I_2 \cup \{x\}$. These sets are clearly independent and of the correct size (n); and the partition is unique since the partition of T - {e,x} into I_1 and I_2 is unique and we have assigned e and x to sets in the only way possible.

Even though Theorems 2.2 and 2.6 cannot be extended to arbitrary graphs (figure 5 (a) shows a graph which is bivariegated but whose LMIS has fewer than half as many vertices as G, and in figure 5 (b) the graph is not bivariegated but contains a LMIS of the desired size), the Theorem 2.2 can be extended to include a certain class of graphs.

(a)   (b)

Figure 5

Definition 2.11: A graph G is called <u>unicyclic</u> if G contains precisely one cycle. The graph in figure 6 is unicyclic.
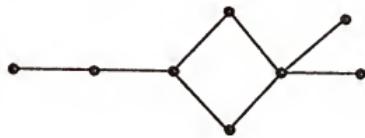
Figure 6

Theorem 2.12: Let G be a connected unicyclic graph with 2n vertices such that G consists of a tree T and a cycle of length 4k which is joined by an additional edge to any point of the tree. Then G is bivariegated if and only if the largest maximal independent set of G has cardinality n.

Proof: First notice that a cycle of 4k points is bivariegated: Label the points of the cycle v_1, v_2, ...v_{4k}; then put $v_1 \in V_1$, $v_2 \in V_1$, v_3, $v_4 \in V_2$, ... v_{4k-4}, $v_{4k-3} \in V_1$, v_{4k-1}, $v_{4k} \in V_2$ to get the desired partition. Also notice that the cycle's LMIS consists of 2k points -- either the points with even labels or the points with odd labels. Thus the statement of the theorem is equivalent to saying that G is

bivariegated if and only if T is, so that the result follows immediately from the result for trees.

Figure 7 shows that if G is unicyclic and has a 1-factor, it is not necessarily true that G has a cycle of size 4k.

Figure 7

However, if G is unicyclic with cycle of length 4k and has a 1-factor, then G is also bivariegated.

Proposition 2.13: A unicyclic graph G with a 1-factor is a bivariegated graph if its cycle C has length $4n$, $n \geq 1$.

Proof: It is sufficient to notice that G - C consists of trees for which having a 1-factor is equivalent to being bivariegated. The two are also equivalent for a cycle of length 4k. Thus the edges of the 1-factor will be the lines defining the bivariegation. The map will certainly be a bijection since no two edges of the 1-factor are incident, and since no point can be adjacent to two points of the set of the partition not containing it (else there would be an additional cycle).

In general we can state the following:

Theorem 2.14: If G_1 and G_2 both are bivariegated (have 1-factors), and if G is the graph formed by G_1, G_2 and an additional edge joining

G_1 to G_2, then G also is bivariegated (has a 1-factor). Conversely, if G is $G_1 \cup G_2 \cup \{g_1, g_2\}$ where $g_1 \in G_1$ and $g_2 \in G_2$, and if G and G_1 are both bivariegated (have 1-factors) then so is (has) G_2.

Proof: If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ then let the bivariegation partitions be $\{V_{11}, V_{12}\}$ and $\{V_{21}, V_{22}\}$ respectively. If the additional edge joins a point of V_{1j} to a point of $V_{2\ell}$, j = 1, 2, ℓ = 1, 2, then the bivariegation for G will be $\{V_{1j} \cup V_{2\ell}, (G_1 \backslash V_{1j}) \cup (G_2 \ V_{2\ell})\}$.

The 1-factor of G will simply be the union of the 1-factors of G_1 and G_2.

The converse is proved analogously.

CHAPTER III

INDEPENDENT SETS AND STRUCTURE

Counting Independent Sets

One of the goals of most areas of mathematics is to completely characterize the constructs or objects under consideration. In graph theory it is no different, and at the present there is no complete set of invariants for graphs or even for trees; that is, there is no sequence of numbers that completely determines what the graph or tree is.

Since it was observed that independent sets have a lot to do with determining whether a tree has a 1-factor, it was conjectured that independent sets might somehow determine a tree's structure more completely. The conjecture was enhanced by the fact that there exist computer programs that list all maximal independent sets of a given tree [3].

For a tree T, $|T| = n$, define a sequence $\sigma(T) = (\xi_1, \xi_2, \ldots, \xi_{n-1}; \beta)$ where ξ_i is the number of maximal independent set of T of size i, and β is the number of end vertices of T. The results of a computer program showed that this sequence does uniquely determine a tree for $n = 1, 2, \ldots, 10$. This chapter is the record of what was discovered while investigating whether $\sigma(T)$ completely determines T for $n > 10$. (The first counterexample known is for $n = 16$; see Appendix I for others.)

One of the first things noticed about independent sets and structure was that if T has precisely two maximal independent sets then T must be a star, as in figure 8.

Figure 8

The set consisting of the endpoints forms one maximal independent set, while the "center point," the non-endpoint, is the other maximal independent set. For a star T, $|T| = n$, $\sigma(T) = (1,0,0,\ldots,0,1;n-1)$.

If T has exactly three MIS's, then T must consist of an edge $\{a,b\}$ each end of which is adjacent to some endpoints, as in figure 9.
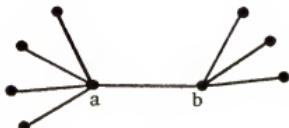
Figure 9

Let A be the set of endpoints adjacent to a, and B be the set of endpoints adjacent to b. Then the maximal independent sets are $A \cup \{b\}$, $B \cup \{a\}$, and $A \cup B$. The sequence $\sigma(T)$ will have three 1's: $\xi_i = 1$ when $i = |A|, |B|, n-2$, and $\beta = n-2$.

If T has more than three MIS's, however, there are many possible forms for T.

In order to study the maximal independent sets of trees, it is helpful to be able to count the total number of MIS's in a given tree.

Assume T is a tree with n points, and sequence $\sigma(T) =$ $(\xi_1, \xi_2, \ldots, \xi_{n-1}; \beta)$. Define $m_T = \sum_{i=1}^{n-1} \xi_i$.

<u>Lemma 3.1</u>: Consider trees with vertices only of degree 1 or 2. Let μ_r be the total number of maximal independent sets of such a tree with r vertices. Then

      (i) $\mu_n = \mu_{n-1} + \mu_{n-5}$ for $n \geq 6$

      (ii) $\mu_n = \mu_{n-2} + \mu_{n-3}$ for $n \geq 4$.

<u>Proof</u>: $\mu_1 = 1$, $\mu_2 = \mu_3 = 2$, $\mu_4 = 3$, $\mu_5 = 4$, $\mu_6 = 5$ by inspection. Note that μ_k is equal to the number of MIS's containing a given endpoint plus the number of MIS's containing the point adjacent to the given endpoint, since all MIS's must contain one of those two points.

    (i) Label the vertices of T, $|T| = n$ by $v_1, v_2, \ldots, v_n$, such that v_i adj v_{i+1} and v_i adj v_{i-1} for $2 \leq i \leq n-1$. $T - \{v_n\}$ has μ_{n-1} maximal independent sets; of these, v_n may be added to those containing v_{n-2} so that T has at least μ_{n-1} MIS's. How many additional sets are created? The new MIS's are those containing v_{n-3} and v_n but neither v_{n-2} nor v_{n-1}. The number of sets containing v_{n-3} equals the number of MIS's containing v_{n-5} plus the number of MIS's containing v_{n-6}, but this is exactly μ_{n-5}. Thus $\mu_n = \mu_{n-1} + \mu_{n-5}$, $n \geq 6$.

    (ii) Label the vertices of T by $v_1, v_2, \ldots, v_n$ from right to left:

Figure 10

Associate with each vertex v_k a number x_k which is the number of MIS's to which v_k belongs with respect to the tree containing only $v_1, \ldots, v_k$.

Then for $k \geq 4$, $x_k = x_{k-2} + x_{k-3}$ since if v_k is in a MIS, then v_{k-1} cannot be, but either v_{k-2} or v_{k-3} must be. Thus $x_k = \mu_{k-2}$ and $\mu_n = x_n + x_{n-1} = \mu_{n-2} + \mu_{n-3}$. $(x_1 = x_2 = x_3 = 1, x_4 = 2)$.

<u>Corollary 1 to Lemma 3.1(i)</u>: Let T be a tree. Then diam $T \leq m_T$.

<u>Proof</u>: Let α be a diametral path of T. As a tree in its own right, α has μ_α maximal independent sets. But each MIS of α must be included in a MIS of T. Thus $\mu_\alpha \leq \mu_T$. Consider only trees with vertices of degree 1 or 2 to show that diam $T \leq \mu_\alpha$. Let T_n be such a tree with n vertices, and proceed by induction on diam T.

diam $T_n = \mu_{T_n}$ for diam $T_n = 3,4,5$.

Assume diam $T_k \leq \mu_T$ for diam $T_k < n$, $n \geq 6$, and let diam $T_k = n$; i.e., $k = n + 1$.

By Lemma 3.1, $\mu_{T_{n+1}} = \mu_{T_n} + \mu_{T_{n-4}}$ and

(*) $\mu_{T_{n-4}} \geq 1$ for $n \geq 5$,

so diam $T_{n+1} = $ diam $T_n + 1 \leq \mu_{T_n} + \mu_{T_{n-4}} = \mu_T$ by the induction hypothesis and (*).

<u>Corollary 2 to Lemma 3.1(i)</u>: If T is a tree and E is the set of end-points of T, then $\mu_{T-E} < \mu_T$.

<u>Proof</u>: diam $T-E = $ (diam) $T-2$ since a diametral path contains two endpoints. Thus $|\alpha_{T-E}| = |\alpha_T| - 2$ where α denotes diametral path. We must add two endpoints to the diametral path of α_{T-E} to get α_T. But as shown in Corollary 1, by adding one point to a diametral path, we get at least one more MIS. Thus by adding two endpoints, will well have added <u>at least</u> 1 to μ_{T-E}. Thus $\mu_{T-E} < \mu_T$.

But note that we do not always add 2 to μ_{T-E}:
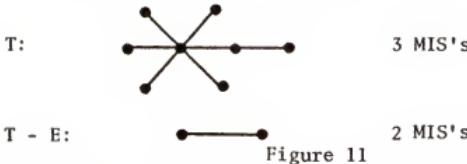
Figure 11

Let T be a tree, $|T| = n$, with 3 end vertices such that there is one vertex v_p with degree 3, and all other vertices have degree 1 or 2, and at least one of the end vertices is adjacent to v_p. Figure 12 shows two examples of such trees.
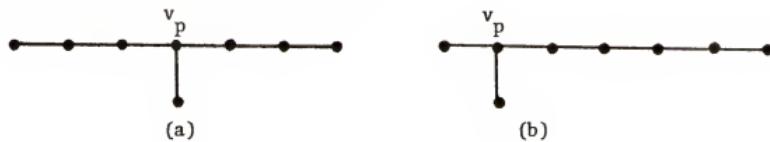
Figure 12

If, as in figure 12 (b), there are two end vertices adjacent to v_p, then $m_T = \mu_{n-1}$ since the two end vertices are in exactly the same maximal independent sets, so that removal of one of them does not change the total number of maximal independent sets in T. If v_p is adjacent to only one vertex, m_T can be expressed in two ways. Label the vertices of the diametral path by $v_1, \ldots, v_{n-1}$.

(1) Label the third end vertex v_k' if it is adjacent to the point v_k, and let x_k' = number of MIS's containing v_k' with respect to

$$x_k = x_{k-2} + x_{k-3} \text{ as before, and}$$
$$x_k' = x_{k-1} + x_{k-2}$$
$$x_{k+1} = x_k'$$
$$x_{k+2} = x_k + x_k'. \text{ Now continue as before, and } m_T = x_{n-1} + x_{n-2}.$$

(2) Again, label the third end vertex v_k' if it is adjacent to v_k. Define $\lambda(e)$ = number of MIS's of T containing e. Then $m_T = \lambda(v_k) + \lambda(v_k')$. There are now two paths of length greater than one from vertex v_k to endpoints. One has length k-1 (with k-1 points other than v_k) and the other has length n-k (with n-k vertices other than v_k) -- see figure 13.
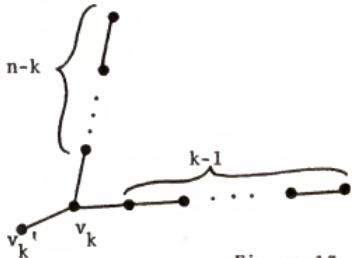
Figure 13

$\lambda(v_k)$ = (number of MIS's v_k is in in path of length k-1) ×

(number of MIS's v_k is in in path of length n-k)

$$= \mu_{k-2} \cdot \mu_{n-k-1},$$

and similarly, $\lambda(v_k') = \mu_{k-1} \cdot \mu_{n-k}$

so that $m_T = \mu_{k-2}\mu_{n-k-1} + \mu_{k-1}\mu_{n-k}$.

What happens if two additional end vertices are added to a path of length n, to distinct points which are not already adjacent to end vertices? (If the two end vertices were added to the same point, m_T could be counted as though only one end vertex was added since the two would be in all the same MIS's.) Suppose they are added to adjacent points as in figure 14.
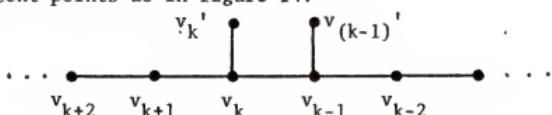
Figure 14

$x_k, x_{k-1}, x_{(k-1)}'$ are found as before, but

$$x_k' = x_{(k-1)}' + x_{k-1}$$

$$x_{k+1} = x_k'$$

$x_{k+2} = x_k + x_k'$; then proceed as usual.

The other case to consider is as in figure 15, where the new points are added to points connected by a path of length 2.
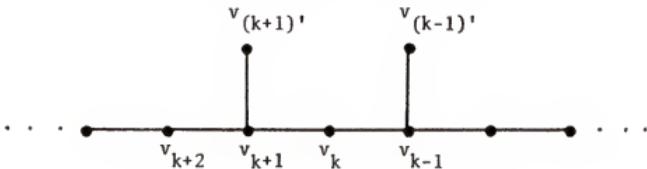
Figure 15

Again, x_k, $x_{(k-1)}'$, and x_{k-1} are found as before, and

$$x_{(k+1)}' = x_k + x_{k-1} = x_{(k-1)}' + x_{k-1}$$

$x_{k+1} = x_{k-1} + x_{(k-1)}'$, and then proceed as in previous cases.

If the new endpoints are added to points at a distance greater than 2 from each other, we may treat each one as if it had been added singly. And if more than two new endpoints are added, we can refer to the above cases and consider them in pairs or singly, as necessary.

Proposition 3.2: Let T be a tree which is a path of length n - 1; i.e., $|T| = n$. Let T' be a tree which is a path of length n - 2 with one additional vertex v_p' of degree 3 and three end vertices (so $|T'| = n$). Then $\mu_n \geq m_T$.

Proof: Label the vertices of T and T' from right to left by $v_1, v_2, \ldots, v_n$ and $v_1', v_2', \ldots, v_n'$, respectively, such that if $v_p' = v_{k-1}'$, the end vertex adjacent to v_p' is v_k' and v_p' is also adjacent to v_{k-2}' and v_{k+1}'

(see figure 16). Now associate with v_i (v_i') a number x_i (x_i') where x_i (x_i') is the number of MIS's containing v_i (v_i') in the tree consisting of the vertices previously labeled by x_j's (x_j''s) starting with $j = 1$. Then $x_{k+1} = x_{k-1} + x_{k-2} \geq x_{k-2}' + x_{k-3}' = x_k' = x_{k+1}'$; and since the labeling continues, from this point, by the same process in both trees, and for the same number of points,

$$m_{T'} = x_n' + x_{n-1}' \leq x_n + x_{n-1} = m_T = \mu_n.$$
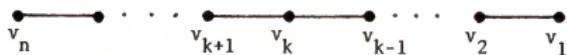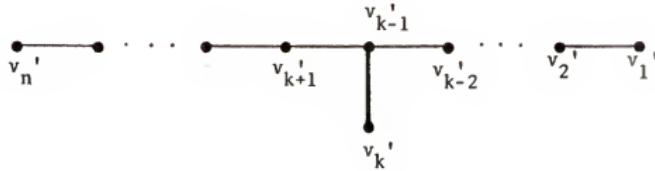
Figure 16

There are two additional facts about trees which are paths and their maximal independent sets.

Proposition 3.3: If T is a tree which is a path of length 2n, then in $\sigma(T) = (\xi_1, \ldots, \xi_{n-1}; 2)$, $\xi_k = 0$ when $k < \frac{2n}{3}$.

Proof: Since every point of T must be a member of a given MIS or adjacent to a member of the set, then the smallest set obtainable is the one with two vertices between every two vertices of the MIS, so that the MIS must contain at least a third of the points of T. More precisely, we have

Case 1: $|T| = 2n = 3r + 1$. $\xi_{r+1} \neq 0$ and has the smallest index of the nonzero ξ_i. $r+1 \geq n-2$ when $n \leq 8$, but $r+1 > n-2$ otherwise. However, $r+1 = \frac{2n-1}{3} + 1 > \frac{2n}{3}$ for all n.

Case 2: $|T| = 2n = 3r$. $\xi_r = \xi_{\frac{2n}{3}} \neq 0$ and has the smallest index of the nonzero ξ_i.

Case 3: $|T| = 2n = 3r + 2$. $\xi_{r+1} \neq 0$ and has the smallest index of the nonzero ξ_i.

$$r+1 = \frac{2n-2}{3} + 1 > \frac{2n}{3} \text{ for all } n.$$

$r+1 \geq n-2$ when $n \leq 7$, but $r+1 < n-2$ otherwise. Thus $\xi_k = 0$ for $k < \frac{2n}{3}$.

Note: If $|T| = 2n$ and T has a 1-factor, then $\xi_{n+1} = \xi_{n+2} = \ldots = \xi_{2n-1} = 0$ by Theorem 2.6.

Proposition 3.4: Let T be a tree, $|T| = n$, where all vertices have degree 1 or 2. Then $m_T = \mu_n \leq 2^{[\frac{n}{2}]}$. (Where $[a]$ = greatest integer less than or equal to a.)

Proof: The inequality holds for $n = 1,2,3,4$. Assume it is true for all $k < n$ and let T have n vertices.

$\mu_n = \mu_{n-2} + \mu_{n-3} \leq 2^{[\frac{n-2}{2}]} + 2^{[\frac{n-3}{2}]}$ by the inductive hypotheses; so if n is even,

$$\mu_n \leq 2^{\frac{n-2}{2}} + 2^{\frac{n-2}{2}-1} \leq \frac{2^{\frac{n-2}{2}} \cdot 3}{2} \leq 2^{\frac{n-2}{2}+1} = 2^{[\frac{n}{2}]}.$$

If n is odd, then

$$\mu_n \leq 2^{\frac{n-3}{2}} + 2^{\frac{n-3}{2}} = 2^{\frac{n-1}{2}} = 2^{[\frac{n}{2}]}.$$

Corollary to Proposition 3.4: If n is even, and if T is a tree as in Proposition 3.4.

$$\mu_n \leq 2^{\frac{n}{2} - 1}.$$

The proof is analogous to that of the proposition.

Some Examples and Comparisons

Let T be a tree, and $v \in T$ be a vertex of T. Define m(v) to be the number of maximal independent sets of T containing v; m(u,v) will be the number of MIS's of T containing u and v; m(u,~v) is the number containing u but **not** v, and so on. If the tree in question is not clear from the context, $m_T(v)$ or $m_T(u,v)$ will be used for clarity.

<u>Example 3.5</u>: If T is a tree as in figure 17, where $|T| = 1+p_1+p_2+\ldots+p_k$, then

$$m_T = m(c) + \sum_{i=1}^{k} m(a_i) - \sum_{\substack{i \neq j \\ i<j}} m(a_i,a_j)$$

$$+ \ldots + (-1)^{k-1} m(a_1,a_2,\ldots,a_k)$$

$$= \mu_{p_1-1} \mu_{p_2-1} \cdots \mu_{p_k-1}$$

$$+ (\mu_{p_1-2} \mu_{p_2} \cdots \mu_{p_k} + \mu_{p_1} \mu_{p_2-2} \cdots \mu_{p_k} + \ldots \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \begin{array}{l} \text{k of} \\ \text{these} \end{array}$$

$$+ \mu_{p_1} \mu_{p_2} \cdots \mu_{p_{k-1}} \mu_{p_k-2})$$

$$- \sum_{\substack{i \neq j \\ i<j}} \mu_{p_i-2} \mu_{p_j-2} \prod_{\substack{\ell=1 \\ \ell \neq i,j}}^{k} \mu_{p_\ell}$$

$$+ \ldots$$

$$+ (-1)^{k-1} (\mu_{p_1-2} \mu_{p_2-2} \cdots \mu_{p_k-2})$$

Figure 17

Example 3.6:

T:
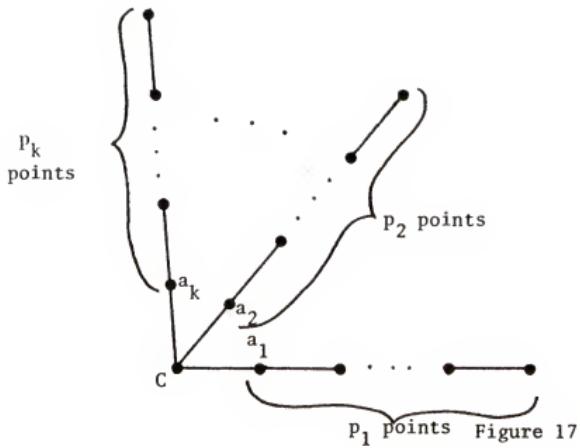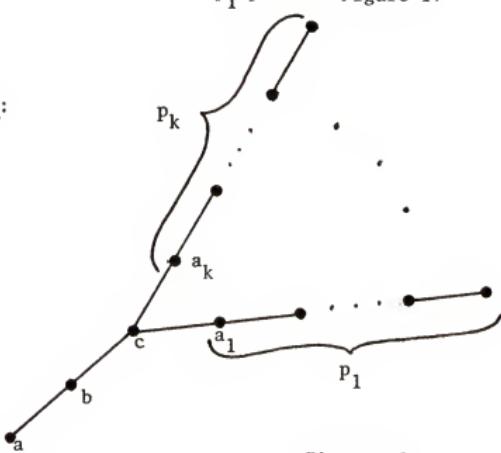
Figure 18

$$m_T = m(a) + m(b)$$

$$= \text{(total from Example 3.5)} +$$

$$\mu_{p_1} \mu_{p_2} \cdots \mu_{p_k} .$$

Example 3.7:
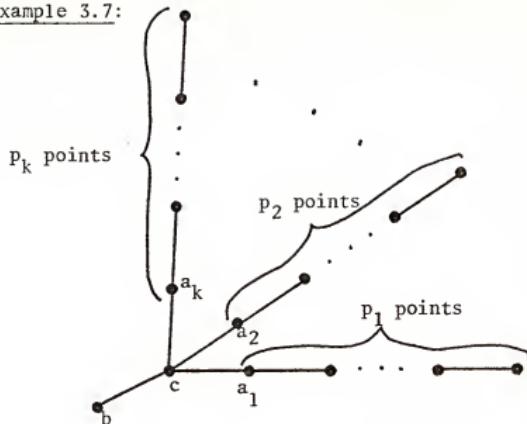
Figure 19

The tree T in figure 19 has

$$m_T = m(c) + m(b)$$

$$= \mu_{p_1-1}\mu_{p_2-1}\cdots\mu_{p_k-1} + \mu_{p_1}\mu_{p_2}\cdots\mu_{p_k}$$

where $p_i \geq 2$ with at least one $p_i > 2$.

If T_1 is a tree like the one in figure 17 and T_2 is a tree like the one in figure 19, with $|T_2| = |T_1| + 1$, then $m_{T_2} \geq m_{T_1}$. (Since the only difference between T_1 and T_2 is that T_2 has an additional point, then T_2 must have at least as many MIS's as T_1.)

Lemma 3.8: Let T be a tree like the one in Example 3.7. Then $m_T \leq 2^{\left[\frac{r+1}{2}\right]}$ where $p_1 + p_2 + \ldots + p_k = r$, $k \geq 2$.

Proof: Recall from Corollary to Proposition 3.4 that if n is even, $\mu_n \leq 2^{\frac{n}{2}-1}$

$$m_T = \mu_{p_1-1}\mu_{p_2-1}\cdots\mu_{p_k-1} + \mu_{p_1}\mu_{p_2}\cdots\mu_{p_k}$$

(if k is 1, this is just the corollary).

Case 1: All p_i, $i = 1, \ldots, k$, are even.

Then
$$m_T \leq 2^{[\frac{p_1-1}{2}]} \; 2^{[\frac{p_2-1}{2}]} \; \ldots \; 2^{[\frac{p_k-1}{2}]} + 2^{\frac{p_1}{2}-1} \cdot 2^{\frac{p_2}{2}-1} \; \ldots \; 2^{\frac{p_k}{2}-1}$$

$$= 2^{\frac{p_1}{2}-1} \; 2^{\frac{p_2}{2}-1} \; \ldots \; 2^{\frac{p_k}{2}-1} + 2^{\frac{p_1}{2}-1} \; \ldots \; 2^{\frac{p_k}{2}-1}$$

$$= 2^{\frac{r}{2}-k} + 2^{\frac{r}{2}-k} = 2^{\frac{r}{2}-k+1} \leq 2^{[\frac{r+1}{2}]}$$

by Proposition 3.4, its corollary, and the fact that $k \geq 2$.

Case 2: All p_i, $i = 1, \ldots, k$ are odd.

(Then all $p_i - 1$ are even and $[\frac{p_i}{2}] = \frac{p_i-1}{2}$)
$$m_T \leq 2^{\frac{p_1-1}{2}} \; \ldots \; 2^{\frac{p_k-1}{2}} + 2^{\frac{p_1-1}{2}} \; \ldots \; 2^{\frac{p_k-1}{2}}$$

$$= 2^{\frac{r}{2}-\frac{k}{2}} + 2^{\frac{r}{2}-\frac{k}{2}}$$

$$= 2^{\frac{r}{2}-\frac{k}{2}+1} \leq 2^{[\frac{r+1}{2}]}.$$

Case 3: Some p_i, say p_1, is even, and some p_i, say p_2, is odd.

Then $\mu_{p_1} \leq 2^{\frac{p_1}{2}-1}$, $\mu_{p_2-1} \leq 2^{\frac{p_2-1}{2}-1}$

and $[\frac{p_1-1}{2}] = \frac{p_1-1}{2} - \frac{1}{2}$ and $[\frac{p_2}{2}] = \frac{p_2-1}{2}$

Thus
$$m_T \leq 2^{\frac{r}{2}-\frac{k}{2}-\frac{3}{2}} + 2^{\frac{r}{2}-\frac{3}{2}}$$

$$\leq 2 \cdot 2^{\frac{r}{2}-\frac{3}{2}} = 2^{\frac{r}{2}-\frac{3}{2}+1} = 2^{\frac{r-1}{2}} \leq 2^{[\frac{r+1}{2}]}.$$

Lemma 3.9: Let $y = 2^x + 2^{k-x}$, k an integer, $1 \leq x \leq k-1$. Then the maximum value of y occurs when $x = k-1$ or $x = 1$.

<u>Proof</u>: $y = 2^x + 2^{k-x}$, $1 \le x \le k - 1$

Maximize this.

$$y = e^{x \log 2} + e^{(k-x)\log 2}$$

$$y' = (\log 2)e^{x \log 2} - (\log 2)e^{(k-x)\log 2}$$

Set this equal to 0.

$$(\log 2)e^{x \log 2} = (\log 2)e^{(k-x)\log 2}$$

$$\Rightarrow \qquad x = k - x$$

$$x = \frac{k}{2}$$

but $y'' = (\log 2)^2 e^{x \log 2} + (\log 2)^2 e^{(k-x)\log 2} > 0$

so this is a minimum.

Therefore $x = 1$ or $x = k-1$ gives a maximum value for y.

<u>Lemma 3.10</u>: $2^{k-1} 2^{[\frac{k_3-4}{2}]} + 2^{[\frac{k_3-1}{2}]} \le 2^{k+[\frac{k_3}{2}]}$; $k > 3$, $k_3 > 4$

<u>Proof</u>: $2^{k-3} + 1 \le 2^{k-2}$

$$\Rightarrow \quad 2^{k-1} + 2^2 \le 2^k \le 2^{k+[\frac{k_3}{2}]} - [\frac{k_3-4}{2}]$$

$$\Rightarrow \quad 2^{k-1} 2^{[\frac{k_3-4}{2}]} + 2^{[\frac{k_3-1}{2}]} \le 2^{k+[\frac{k_3}{2}]}$$

since $[\frac{k_3-1}{2}] - [\frac{k_3-4}{2}] = \begin{cases} 1 \text{ if } k_3 \text{ even} \\ 2 \text{ if } k_3 \text{ odd} \end{cases}$

<u>Proposition 3.11</u>: If T is a tree of the following form
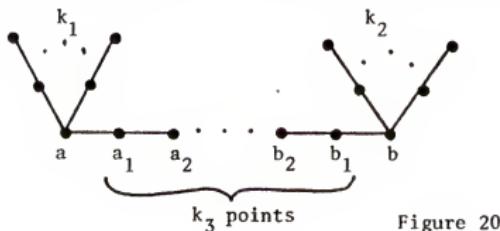
Where $k_3 > 6$
$k_1 + k_2 = k$
$|T| = 2k + k_3 + 2$

Figure 20

then $m_T \leq 2^{k + \left[\frac{k_3 + 2}{2}\right]}$.

(It is also true for $k_3 \leq 6$, as shown in Examples 3.12 through 3.15).

Proof:

$$m_T = m(a,b) + m(a,\sim b) + m(b,\sim a) + m(\sim a,\sim b)$$

$$= m(a,b) + (m(a,b_1) + m(a,b_2))$$

$$\qquad + (m(b,a_1) + m(b,a_2))$$

$$\qquad + (m(a_1,b_1) + m(a_1,b_2) + m(a_2,b_1) + m(a_2,b_2)$$

$$= \mu_{k_3-2} + (\mu_{k_3-3} 2^{k_2} + \mu_{k_3-4}(2^{k_2}-1))$$

$$\qquad + (\mu_{k_3-3} 2^{k_1} + \mu_{k_3-4}(2^{k_1}-1))$$

$$\qquad + (\mu_{k_3-4} \cdot 2^{k_1} 2^{k_2} + \mu_{k_3-5} 2^{k_1}(2^{k_2}-1)$$

$$\qquad + \mu_{k_3-5}(2^{k_1}-1) 2^{k_2} + \mu_{k_3-6}(2^{k_1}-1)(2^{k_2}-1))$$

$$= \mu_{k_3-2} + (2^{k_2} + 2^{k_1}) [\mu_{k_3-3} + \mu_{k_3-4} - \mu_{k_3-5} \mu_{k_3-6}]$$

$$\qquad + 2^k (\mu_{k_3-4} + 2\mu_{k_3-5} + \mu_{k_3-6}) - 2\mu_{k_3-4} + \mu_{k_3-6}$$

Now by using

$$\mu_{k_3-3} + \mu_{k_3-4} = \mu_{k_3-1}$$

$$\mu_{k_3-5} + \mu_{k_3-6} = \mu_{k_3-3}$$

$$\mu_{k_3-1} - \mu_{k_3-3} = \mu_{k_3-4}$$

$$\mu_{k_3-4} + \mu_{k_3-5} = \mu_{k_3-2}$$

$$\mu_{k_3-2} + \mu_{k_3-3} = \mu_{k_3}$$

$$\mu_{k_3-2} + \mu_{k_3-6} = \mu_{k_3-1}$$

(all by Lemma 3.1)

we find that

$$m_T = (2^{k_2} + 2^{k_1})\mu_{k_3-4} + 2^k\mu_{k_3} - 2\mu_{k_3-4} + \mu_{k_3-1}$$

$$\leq 2^{k-1}\mu_{k_3-4} + 2\mu_{k_3-4} + 2^k\mu_{k_3} - 2\mu_{k_3-4} + \mu_{k_3-1}$$

by Lemma 3.8

$$\leq 2^{k-1} \cdot 2^{[\frac{k_3-4}{2}]} + 2^k 2^{[\frac{k_3}{2}]} + 2^{[\frac{k_3-1}{2}]}$$

by Proposition 3.4.

$$\leq 2^{k+[\frac{k_3}{2}]} + 2^{k+[\frac{k_3}{2}]} = 2^{k+1+[\frac{k_3}{2}]} = 2^{k+[\frac{k_3+2}{2}]}$$

by Lemma 3.10.

The following examples show Proposition 3.10 is true for $\sim$, $||$

$k_3 = 1$, 4, 5, 6 ($k_3 = 2$, 3 are similar).

Example 3.12:
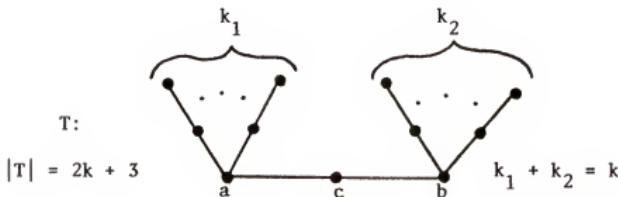
Figure 21

$$m(a,b) = 1$$
$$m(a) = 2^{k_2}$$
$$m(c) = 2^k$$
$$m(b) = 2^{k_1}$$
so $$m_T = 1 + 2^k + 2^{k_2} + 2^{k_1}$$

<u>Claim:</u> $1 + 2^k + 2^{k_2} + 2^{k_1} \le 2^{k+1}$

or $1 + 2^{k_2} + 2^{k_1} \le 2^k$

since $2^k + 2^k = 2 \cdot 2^k = 2^{k+1} \Rightarrow 2^{k+1} - 2^k = 2^k$

<u>Proof:</u> $(2^k - 1) = (2 - 1)(1 + 2 + 2^2 + \ldots + 2^{k-1}) \ge 2^{k_1} + 2^{k_2}$

where $k_1 + k_2 = k$, k_1, $k_2 > 0$, therefore

$1 + 2^{k_1} + 2^{k_2} \le 2^k$ so claim is proved.

T_1:   Note that T_1 has 2^{k+1} MIS's and $2k + 3$ points.
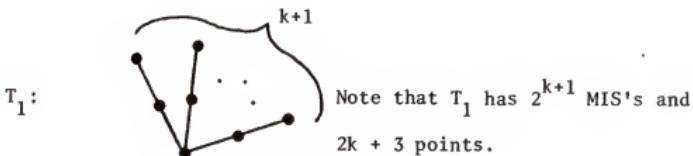
Figure 22

<u>Example 3.13:</u>

T: $|T| = 2k + 6$

$k_3 = 4$

Figure 23

T_1: $m_{T_1} = 2^{k+2} + 1$

$|T_1| = 2k + 6$
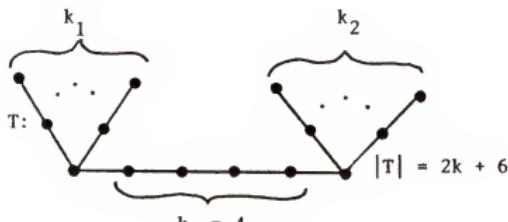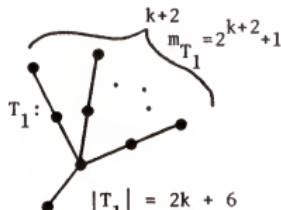
Figure 24

$m(a) = 1 \cdot 2^{k_2} + (2^{k_2} - 1)$

$m(b) = 2^{k_1} + (2^{k_1} - 1)$

$m(a,b) = 2$

$m(\sim a, \sim b) = 2^k + 2^{k_1}(2^k - 1) + 2^{k_2}(2^{k_1} - 1) + (2^{k_2} - 1)(2^{k_1} - 1)$

$$= 2^k + 2^k - 2^{k_1} + 2^k - 2^{k_2} + 2^k - 2^{k_2} - 2^{k_1} + 1$$

$$m_T = 2^{k_2} + 2^{k_2} - 1 + 2^{k_1} + 2^{k_1} - 1 + 2$$

$$+ 4 \cdot 2^k - 2 \cdot 2^{k_1} - 2 \cdot 2^{k_2} + 1$$

$$= 2^{k+2} + 1$$

note that $m_T = m_{T_1}$.

Example 3.14:
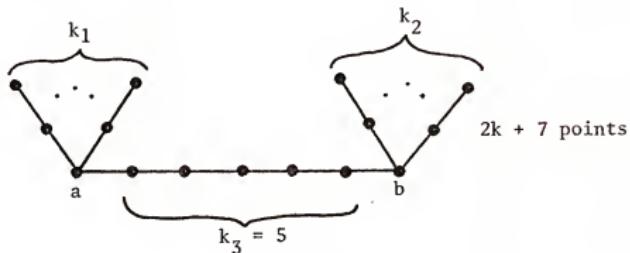
Figure 25

want $m_T \leq 2^{k+3}$

$$m(a) = 2 \cdot 2^{k_1} + 1(2^{k_2} - 1)$$
$$m(b) = 2 \cdot 2^{k_2} + (2^{k_1} - 1)$$

$$m(a,b) = 2$$

$$m(\sim a, \sim b) = 2^{k_1} 2^{k_2} + 2^{k_1}(2^{k_2} - 1) + 2^{k_2}(2^{k_1} - 1) + (2^{k_1} - 1)(2^{k_2} - 1)$$

$$= 2^k + 2^k - 2^{k_1} + 2^k - 2^{k_2} + 2^k - 2^{k_1} - 2^{k_2} + 1$$

$$m_T = 2^{k_2} + 2^{k_1} + 1 + 2^{k+2} \quad (\leq 2^k(1+4) = 2^k \cdot 5 \leq 2^{k+3})$$

$$\leq 2^{k-1} + 2 + 1 + 2^{k+2} \leq 2^{k+3}$$
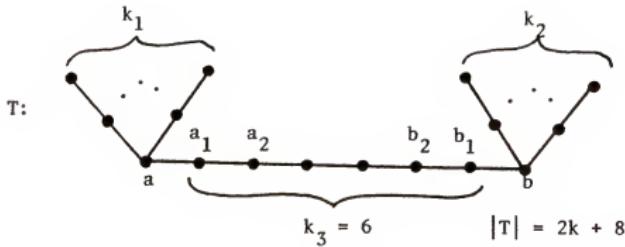
Example 3.15:

Figure 26

Show $m_T \le 2^k(2^3 + 1)$ or $2^k \cdot 2^4$.

$$m(a) = m(a,b_2) + m(a,b_1)$$

$$= \mu_3 \cdot 2^{k_2} + \mu_2 (2^{k_2} - 1) = 2 \cdot 2^{k_2} + 2 \cdot 2^{k_2} - 2$$

$$m(b) = 2 \cdot 2^{k_1} + 2 \cdot 2^{k_1} - 2$$

$$m(a,b) = \mu_4 = 3$$

$$m(\sim a, \sim b) = m(a_1,b_1) + m(a_1,b_2) + m(a_2,b_1) + m(a_2,b_2)$$

$$= 2 \cdot 2^k + 1 \cdot 2^k - 2^{k_1} + 1 \cdot 2^k - 2^{k_2} + 1 \cdot 2^k - 2^{k_1} - 2^{k_2} + 1$$

$$m_T = 2(2^{k_2} + 2^{k_1}) + 5 \cdot 2^k$$

$$\le 2(2^{k-1} + 2) + 5 \cdot 2^k$$

$$= 2 \cdot 2^{k-1} + 4 + 5 \cdot 2^k = 2^k (1 + 5) + 4 = 2^k(6) + 4$$

$$\le 2^{k+3} + 4 \le 2^{k+4}$$

Example 3.16: Let T be a tree such that $T = T_1 \cup T_2$, where $T_1 \cap T_2 = \{a\}$. Let B_1 be the set of points of T_1 that are adjacent to a; and B_2 be the set of points of T_2 that are adjacent to a.

Then

$$m_T = m_1(\sim B_1)\ m_2(\sim B_2) + \overline{m_1(\sim B_1)}\ m_2(B_2)$$

$$+ m_1(B_1)\ m_2(B_2) + m_1(B_1)\ \overline{m_2(\sim B_2)}$$

where for $i = 1, 2$,

$m_i(B_i)$ = number of MIS's in T_i (where T_i includes the point a) containing <u>at least</u> one point adjacent to a.

$m_i(\sim B_i)$ = number of MIS's in T_i containing <u>no</u> point adjacent to a.

$\overline{m_i(\sim B_i)}$ = number of MIS's in $T_i\backslash\{a\}$ containing <u>no</u> point adjacent to a.

notice that $m_{T_i} = m_i(B_i) + m_i(\sim B_i)$.

<u>Proposition 3.17</u>: Let T be the tree in figure 27(a) and let T' be the tree in figure 27(b). Then $m_T \leq m_{T'}$. (The subtree T_2 is the same in both trees.)
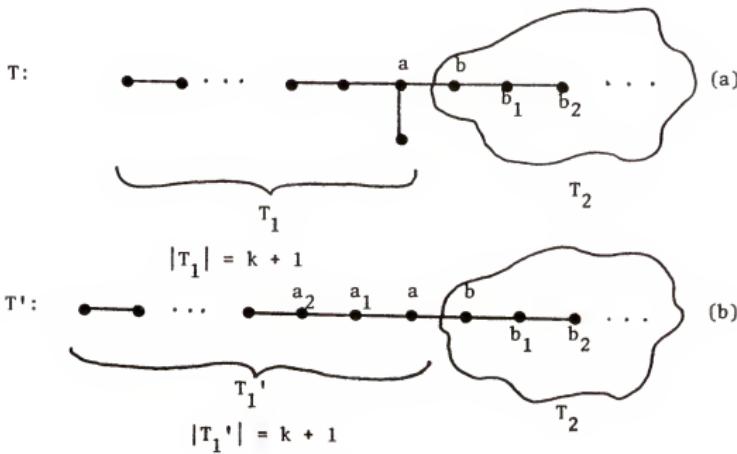
Figure 27

Proof: $m_{T_1 \cup T_2} = m(a,\sim b) + m(\sim a,\sim b) + m(\sim a,b)$

$$= \mu_{k-2}(m_{T_2}(b_1) + m_{T_2}(b_2)) + \mu_{k-1} m_{T_2}(b_1) + \mu_{k-1} m_{T_2}(b)$$

since

$$\mu_{k-2} = m_{T_1}(a) \text{ and}$$

$$\mu_{k-1} = m_{T_1}(\sim a) \text{ by the discussion seen earlier in this chapter,}$$

and since if a is a given MIS then b is not, but either b_1 or b_2 must be.

Similarly,

$$m_{T_1' \cup T_2} = \mu_{k-1}(m_{T_2}(b_1) + m_{T_2}(b_2)) + \mu_{k-2} m_{T_2}(b_1) + \mu_k m_{T_2}(b).$$

But $\mu_{k-1} m_{T_2}(b_2) \geq \mu_{k-2} m_{T_2}(b_2)$

and $\mu_k m_{T_2}(b) \geq \mu_{k-1} m_{T_2}(b)$

implies that

$$\mu_{k-1} m_{T_2}(b_1) + \mu_{k-1} m_{T_2}(b_2) + \mu_{k-2} m_{T_2}(b_1) + \mu_k m_{T_2}(b)$$

$$\geq \mu_{k-2} m_{T_2}(b_1) + \mu_{k-2} m_{T_2}(b_2) + \mu_{k-1} m_{T_2}(b_1) + \mu_{k-1} m_{T_2}(b)$$

and so $m_{T_1 \cup T_2} \leq m_{T_1' \cup T_2}$.

If deg b > 2, then interpret $m_{T_2}(b_1)$ as the number of MIS's containing at least one point of T_2 adjacent to b, and $m_{T_2}(b_2)$ as the number of MIS's of T_2 containing no point adjacent to b.

Figure 28 shows T and T'; the additional labels near points in T_1 and T_1' indicate the number of MIS's the points are in, with respect to T_1 or T_1'.
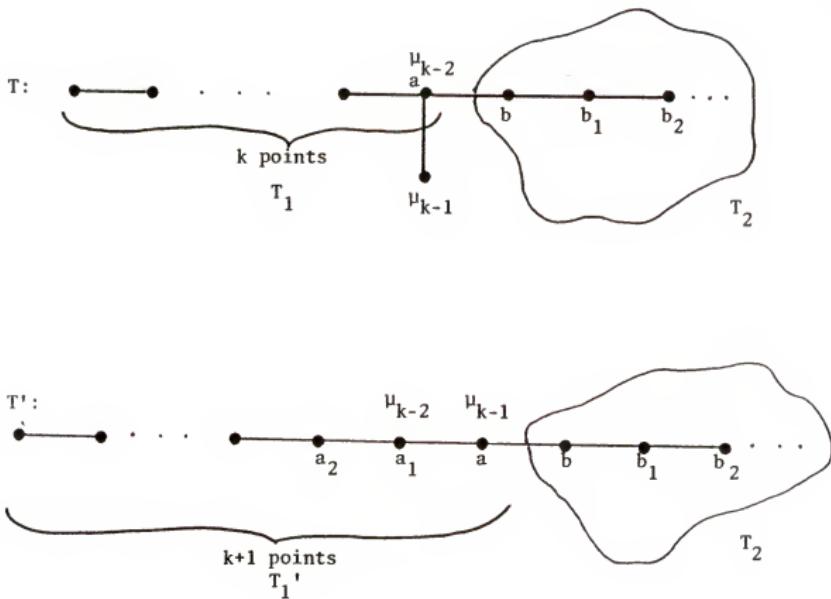
Figure 28

Corollary to Proposition 3.17: A tree which is simply a path of length $n + j$ has more maximal independent sets than a path of length n with j additional end vertices; $j \leq n - 2$.

Proof: Repeatedly apply Proposition 3.17.

Proposition 3.18: Let T be a tree as in figure 29. Then $m_T \leq 2^{k+[\frac{r}{2}]}$ if r is odd, and $m_T \leq 2^{k+[\frac{r}{2}]-1} + 1$ if r is even.

Proof: $m_T = m(a) + m(\sim a) = \mu_{r-1} + 2^k \mu_{r-2} + (2^k - 1)\mu_{r-3}$

$\qquad = \mu_{r-1} - \mu_r + 2^k(\mu_{r-2} + \mu_{r-3}) \leq 2^k(\mu_{r-2} + \mu_{r-3}).$

Case 1: r is odd.

By Proposition 3.4, $m_T \leq 2^k(2^{[\frac{r-2}{2}]} + 2^{[\frac{r-3}{2}]})$

$$\leq 2^{k+1} \, 2^{[\frac{r-2}{2}]} = 2^{k+[\frac{r}{2}]}$$

Case 2: r is even.

By the corollary to Proposition 3.4, and since $[\frac{r-3}{2}] = [\frac{r-2}{2}] - 1$

$$m_T \leq 2^k(2^{[\frac{r-2}{2}]-1} + 2^{[\frac{r-3}{2}]}) \leq 2^{k+1} \, 2^{[\frac{r-2}{2}]-1} = 2^{k+[\frac{r-2}{2}]} \leq 2^{k+[\frac{r}{2}]} + 1.$$
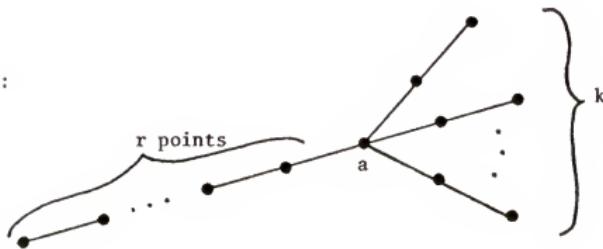
Figure 29

Proposition 3.18 says that the tree in figure 29 has fewer MIS's than the trees in figure 30. $m_{T_1} = 2^{k+[\frac{r}{2}]}$ and $m_{T_2} = 2^{k+[\frac{r}{2}]-1} + 1$.
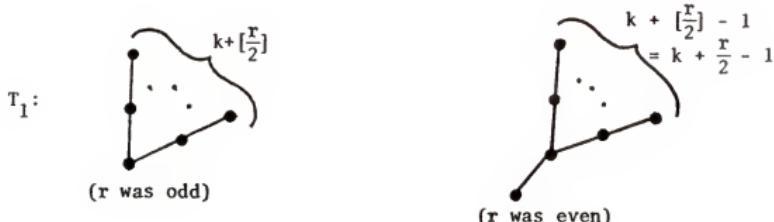
Figure 30

A Smaller Class of Trees

Since the sequence $\sigma(T) = (\xi_1, \xi_2, \ldots, \xi_{n-1}; \beta)$ for tree T, $|T| = n$, was not immediately observed to have any pattern, it was decided that the study should be narrowed to trees with 1-factors, to determine if the sequence completely determined them, and then to generalize if possible. The study was narrowed even further when it was observed that certain sequences have only one nonzero entry besides β.

Consider a tree T, $|T| = 2n$. T has a 1-factor if and only if $\xi_i = 0$ for $i = n+1, n+2, \ldots, 2n-1$, in the sequence $\sigma(T)$. Now suppose that, in addition, $\xi_i = 0$ for $i = 1, 2, \ldots, n-1$; that is, $\sigma(T) = (0, \ldots, 0, \xi_n, 0, \ldots, 0; \beta)$. (Denote this by $\sigma(T) = (\xi_n; \beta)$. Conversely, when $\sigma(T) = (\xi_n; \beta)$ appears, it shall mean that T is a tree with a 1-factor on 2n vertices such that $\xi_i = 0$, $i \neq n$ in $\sigma(T)$). Then we can say something about the structure of T.

Theorem 3.19: Let T, $|T| = 2n$, be a tree with a 1-factor and let $\sigma(T) = (\xi_n; \beta)$. Then $\beta = n$.

Proof: The theorem is true for n = 2. If T is ●—●—●—● , then $\sigma(T) = (3;2)$. Assume it is true for n, and let $|T| = 2n + 2$ and $\xi_1 = \xi_2 = \ldots = \xi_n = 0$. T has at least two remote end vertices. Let e be one of them, with e adj x. Consider T − {e,x}. Suppose T − {e,x} has a MIS of size k, k < n. In T, let y adj x, y ≠ e. If y is in a MIS, M_1, of T − {e,x} of size k, then $M_1 \cup \{e\}$ is a MIS in T of size k + 1, k + 1 < n + 1, contrary to hypothesis. If y is not in a MIS, M_2, of T − {e,x}, then $M_2 \cup \{e\}$ or $M_2 \cup \{x\}$ is a MIS of T of size k + 1 < n + 1, again a contradiction. Hence for T − {e,x}, $\xi_1 = \xi_2 = \ldots = \xi_{n-1} = 0$ so by the inductive hypothesis, T has n endpoints.

Replace $\{e,x\}$ at y. If y has degree larger than 1 in $T - \{e,x\}$, then T will have $n + 1$ endpoints and we are done. So suppose y is an endpoint of $T - \{e,x\}$. Let z adj y and z_1 adj z, $z_1 \neq y$. (z_1 cannot be an endpoint since $T - \{e,x\}$ has a 1-factor -- see Properties (1) and (3) of bivariegated trees.) There is a MIS M of $T - \{e,x\}$ containing y and z_1, and $|M| = n$. Then $(M - \{y\}) \cup \{x\}$ is a MIS of T with only n points, a contradiction. Thus y must be a non-endpoint, or an interior point, so T has $n + 1$ endpoints.

Lemma 3.20: Let T be a tree with a 1-factor, $|T| = 2n$, such that $\sigma(T) = (\xi_n; \beta)$. If a remote end vertex e, e adj x, is joined to an interior point of T, then $\sigma(T \cup \{e,x\}) = (\xi_{n+1}; \beta')$.

Proof: By Theorem 3.19, $\beta = n$ and $\beta' = n + 1$. Let y be a point of T such that deg $y > 1$, where T, e, and x are as in the hypotheses. Every MIS of $T \cup \{e,x\}$ must contain either e or x (but not both), and every MIS of $T \cup \{e,x\}$ must contain a MIS of T (which has size n). Since $\beta = n$, and since T has a 1-factor, every point of T either is an end vertex or is adjacent to an end vertex. Thus when we adjoin $\{e,x\}$ to y, if y is in a MIS M, $M - \{y\} \cup \{e\}$ has size n, but is not maximal since the end vertex which is adjacent to y is neither in $M - \{y\} \cup \{e\}$, nor is it any longer adjacent to a member of the set, which is a contradiction. Thus every MIS of $T \cup \{e,x\}$ has $n + 1$ elements; that is, $(\xi_{n+1}; n+1) = \sigma(T \cup \{e,x\})$.

Theorem 3.21: Let T be a tree with a 1-factor, $|T| = 2n$, such that T has n endpoints. Then T has maximal independent sets only of size n; that is, $\sigma(T) = (\xi_n; n)$.

Proof: The theorem is true for n = 2 (see the proof of Theorem 3.19).
Assume the result is true for n, and let T be a tree with 2n + 2
vertices, a 1-factor, and n + 1 end vertices. Since T has a 1-factor,
T has two remote end vertices, e and e', with adjacent vertices x and
x' respectively, and with x adj y, y ≠ e and x' adj y', y' ≠ e'.
Assume we cannot remove any remote end vertex and obtain a tree with
n end vertices. Then at least one of the remote end vertices, say e,
must be attached to a point that is an endpoint in T - {e,x}. Thus,
in T, deg y = 2. So we have n - 1 end vertices other than e and e'
which cannot be adjacent to e, e', x, x' or y, which leaves 2n + 2 -
(n - 1) - 5 = n - 2 points. However, this implies that two of the
endpoints are adjacent to the same point, which contradicts property
(1) of bivariegated trees (trees with 1-factors). Thus there must be
a point y' with deg y' ≥ 3 such that x' adj y', e' adj x', e' ≠ y',
e' a remote end vertex. If we remove {e',x'} from T, y' is not an
endpoint, and the tree T - {e',x'} has n endpoints, and has a 1-factor.
By the inductive hypothesis, $\sigma(T - \{e',x'\}) = (\xi_n; \beta)$ where $\beta = n$. If
we now replace {e',x'}, T has only maximal independent sets of size
n + 1 by Lemma 3.20; that is, $\sigma(T) = (\xi_{n+1}; n + 1)$.

By combining Theorems 3.19 and 3.21, we have that for T, |T| = 2n,
with a 1-factor, $\sigma(T) = (\xi_n; \beta)$ if and only if $\beta = n$.

Theorem 3.19 and Lemma 3.20 show that a tree whose sequence can
be "built up" from ●—●—●—● by successively adding remote end
vertices to interior points. This building process is not unique.
One such tree on 2n vertices can be obtained by adding a remote end
vertex to two different trees on 2n - 2 vertices. In figure 31, T_1
is obtained from either T_2 or T_3 by adding a remote end vertex to the
indicated points.
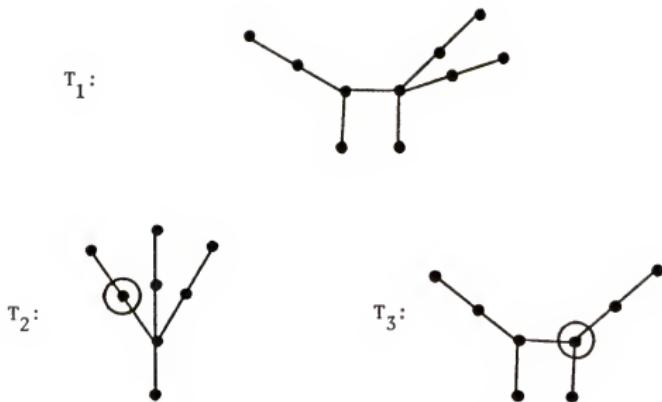
Figure 31

The edges that join the endpoints of a tree T, $\sigma(T) = (\xi_n; n)$, to their adjacent points are precisely the edges of the 1-factor.

If T_1 is a tree with a 1-factor, $|T| = 2n$, all of whose maximal independent sets have n elements, and if we remove the n end vertices of T, then the result T_2 is a tree which does not necessarily have a 1-factor. In fact it could be any one of the trees on n vertices. Notice that the number of remote end vertices of T_1 is precisely the number of end vertices of T_2. Conversely, if we start with an arbitrary tree T_2' on n vertices and to each vertex of T_2' adjoin an end vertex, we then have a tree T_1' which has n end vertices, and which has a 1-factor (the edges of the 1-factor being the new edges) so that $\sigma(T_1') = (\xi_n; n)$.

Definition 3.22: Let T be a tree with n vertices. Let p(T) be the tree obtained by adding one end vertex to each vertex of T. The process of adding these end vertices is called _expansion_; p(T) is the _expanded tree_ of T, and T is the _reduced_ or _core tree_ p(T).

The above discussion shows that there is a one-to-one correspondence between the set of all trees and the set of expanded trees. Since so much information can be found about expanded trees, the relationship between p(T) and T can be expected to give information about T (for example, the number of remote end vertices of p(T) is the number of end vertices of T). The information is often easier to get from p(T) because of its special structure. In addition, if $\sigma(p(T) = (\xi_n;n)$, then ξ_n is the total number of independent sets of T. For if M is a maximal independent set of p(T), then M ∩ T is an independent set of T. Conversely, if I is an independent set of T, then the set consisting of I plus the end vertices of p(T) which are adjacent to points of T - I is a maximal independent set in p(T) of size n. Thus if we are able to count ξ_n, we will then know how to count the total number of independent sets of a tree.

Since we have the one-to-one correspondence between trees and expanded trees, the number of expanded trees on 2n points is equal to the number of trees on n points. Erdös raised the question, "Are there enough integers to be used for the ξ_n's without giving the same integer to two different expanded trees?" (It had been noticed that $\max_{|T|=2n} \xi_n < 2^{n-1} + 1$ -- see Chapter IV.) If there are not enough integers, then the sequence $\sigma(T)$ could not be the complete set of invariants for trees as was conjectured. There are 317,955 trees on 19 vertices [4, p. 232], so there are 317,955 expanded trees on 38 vertices. However, $2^{18} + 1 = 262,144 + 1 < 317,955$. Therefore, the sequence $\sigma(T)$ is not a complete set of invariants for trees. In the appendix are some counterexamples on 16, 18, and 20 points.

The trees in figure 32 show that even if we add to the sequence $\sigma(T)$ the diameter of T, the number of remote end vertices of T, the

sequence of degrees of the vertices, and the number of points in each set of a bipartite partition of T, it is still not enough to completely determine the structure of T!

$$\sigma(T_1) = (0,0,0,1,12,0,0,0,0;4)$$

$$\sigma(T_2) = (0,0,0,5,6,1,0,0,0;4)$$

$$\sigma(p(T_1)) = (160;10)$$

$$\sigma(p(T_2)) = (160;10)$$
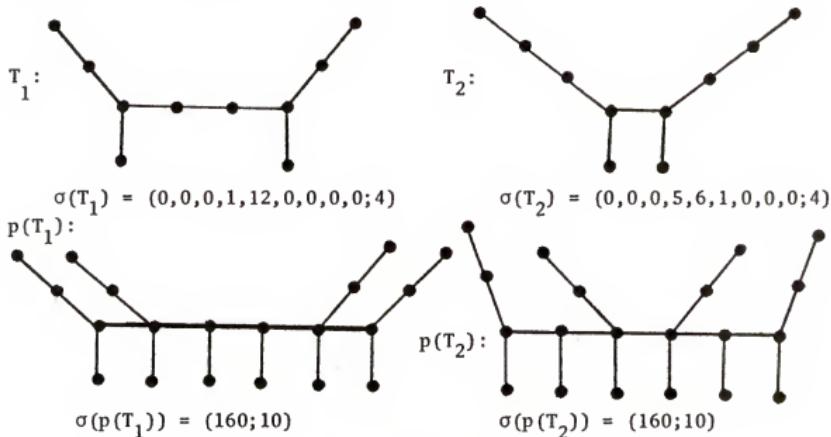
Figure 32

If $\sigma(T) = (\xi_1,\xi_2,\ldots,\xi_{n-1};\beta)$, $|T| = n$, and $\sigma(p(T)) = (\xi_n;n)$, then we can estimate ξ_n using the ξ_i, $i = i,2,\ldots,n-1$.

$$\xi_n \le \sum_{i=0}^{n-1} \text{(number of MIS's of p(T) containing i points of T)}$$

$$= 1 + n + (\xi_2 + \sum_{j>2} \xi_j\binom{j}{2})) + (\xi_3 + \sum_{j>3} \xi_j\binom{j}{3}))$$

$$+ \ldots + (\xi_k + \sum_{j>k} \xi_j\binom{j}{k})) + \ldots + \xi_{n-1}$$

rearranging

$$= 1 + n + \xi_2 + \xi_3 (1 + \binom{3}{2})) + \xi_4 (1 + \binom{4}{2}) + \binom{4}{3}))$$

$$+ \ldots + \xi_k (1 + \binom{k}{2}) + \ldots + \binom{k}{k-1})) + \ldots + \xi_{n-1}(1+\binom{n-1}{2}+\ldots+\binom{n-1}{n-2}))$$

$$= 2^k - k - 1$$

$$= 1 + n + \xi_2 + 4\xi_3 + 11\xi_4 + 26\xi_5 + 57\xi_6 + \ldots + k_{n-1} \, \xi_{n-1}.$$

$$\text{where } k_{n-1} = (1 + \binom{n-1}{2} + \ldots + \binom{n-1}{n-2}))$$

However, this method of counting counts some sets twice. For example, $\xi_n = \xi_6$ for the tree in figure 33 is 23 but the estimate gives $1 + 6 + 2 + 4 + 11 = 24$. The MIS's of T are $\{1,2,5\}$, $\{1,2,4,6\}$, $\{3,5\}$, and $\{3,6\}$.
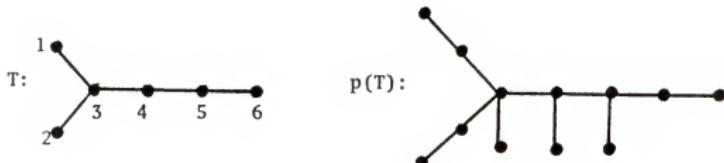
Figure 33

Counting ξ_n; and a Number Theory Bonus

In this section, every tree T will be a tree with a 1-factor such that $\sigma(T) = (\xi_n;n)$. The reduced or core tree of T will be denoted by $p^{-1}(T)$.

For each vertex v in T, define a number $\lambda(v)$ to be the number of maximal independent sets of T which contain v. If v is an interior point, and if w is the endpoint adjacent to v, then $\lambda(v) + \lambda(w) = \xi_n$, since every MIS must contain either v or w. And since the equality holds for every interior point and adjacent endpoint, it follows that $n\xi_n = \sum\limits_{v \in T} \lambda(v)$, or $\xi_n = \dfrac{\sum\limits_{v \in T} \lambda(v)}{n}$. In particular, if e is a remote end vertex and e adj x, then $\lambda(e) + \lambda(x) = \xi_n$. If x adj y, $y \neq e$, then

$\lambda(y) + \lambda(x) = \lambda(e)$ since e belongs to every MIS containing y, and if a MIS M contains x, then $(M-\{x\}) \cup \{e\}$ is also a MIS of the proper size, and these are the only MIS's that could possibly contain e.
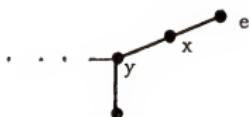
Figure 34

By combining these two equalities we find that $\xi_n = \lambda(y) + 2\lambda(x)$.

If $\sigma(T) = (\xi_n;n)$ and e is a remote end vertex, e adj x, then $\sigma(T-\{e,x\}) = (\xi_{n-1};n-1)$ by Lemma 3.20. If y adj x in T, $y \neq e$, then $\xi_n = 2\xi_{n-1} - \lambda(y)$, since for every MIS M in T - $\{e,x\}$ we have the MIS M $\cup$ $\{e\}$, and the MIS M $\cup$ $\{x\}$, except when M contains y (note that $\lambda(y)$ is the same for T and T - $\{e,x\}$).

The following proposition states some more facts about relationships between λ-numbers.

Proposition 3.23: Let T be an expanded tree with $\sigma(T) = (\xi_n;n)$, that looks like the tree in figure 35; that is, e is a remote end vertex, e adj x, x adj y, $y \neq e$, z_1 is the end vertex adjacent to y, z_2 adj y, $z_2 \neq z_1$, $z_2 \neq x$, and z_3 is the end vertex adjacent to z_2. The structure of the rest of T (which is attached at z_2) does not matter.

T:
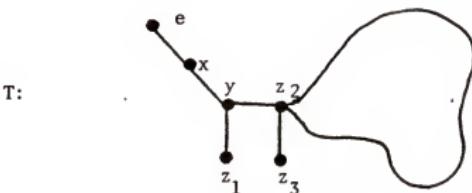
Figure 35

Then

(i) $\lambda(z_3) = 3\lambda(y)$, so that $\lambda(z_3)$ is divisible by 3

(ii) $\lambda(z_1) = 2\lambda(y) + \lambda(z_2)$

(iii) $\lambda(z_2)$ is even

(iv) $\lambda(z_1)$ is even

(v) $\lambda(y)$ and ξ_n have the same parity

(vi) $\lambda(v)$ is independent of the number of remote end vertices attached to v for any $v \in T$ that is an interior point.

In addition,

(vii) Let $\lambda'(v)$ be the number of MIS's containing v in $T - \{e,x\}$, for any $v \in T - \{e,x\}$ and denote $\sigma(T-\{e,x\})$ by $(\xi_{n-1}; n-1)$. Then $\lambda(e) = \xi_{n-1}$.

Proof:

(i) For every MIS M containing y, $z \in M$, $M - \{y\} \cup \{z_1\}$, $M - \{y,e\} \cup \{z_1,x\}$, so $\lambda(z_3) = 3\lambda(y)$, and 3 divides $\lambda(z_3)$.

(ii) This can be proved in two ways:

(a) $\xi_n = \lambda(z_2) + \lambda(z_3) = \lambda(z_2) + 3\lambda(y)$ by (i); $\xi_n = \lambda(y) + \lambda(z_1)$. The difference of these two equations is $0 = 2\lambda(y) + \lambda(z_2) - \lambda(z_1)$ or $\lambda(z_1) = 2\lambda(y) + \lambda(z_2)$

(b) For every MIS M containing y, $z_1 \in M - \{y,e\} \cup \{x,z_1\}$ and $z_1 \in M - \{y\} \cup \{z_1\}$. z_1 is also in every MIS containing z_2.

(iii) Let T_1 be the part of T containing e, x, y, z_1, z_2, z_3 and $T_2 = (T-T_1) \cup \{z_2\}$. $\lambda(z_2) = $ (number of MIS's in T_1 containing z_2) × (number of MIS's in T_2 containing z_2) = 2 × (number of MIS's in T_2 containing z_2). The two sets in T_1 are $\{z_1,z_2,e\}$ and $\{z_1,z_2,x\}$. Thus 2 divides $\lambda(z_2)$.

(iv) follows immediately from (ii) and (iii); and (v) follows
from (iv) and $\xi_n = \lambda(y) + \lambda(z_1)$.

(vi) Let e_1, e_2, ..., e_k be the remote end vertices attached to
some interior point v in T, with e_i adj x_i, x_i adj v,
v = 1, 2, ..., k. Then every MIS containing v must also
contain all the e_i's, and if a MIS contains even one x_i,
then v is <u>not</u> a member of that set. Thus $\lambda(v)$ is not
affected by the size of k.

(vii) $\lambda(e) + \lambda(x) = \xi_n$

and $\lambda(x) + \lambda(y) = \lambda(e)$

so that $2\lambda(e) = \xi_n + \lambda(y)$

so $\lambda(e) = \dfrac{\xi_n + \lambda(y)}{2}$ (1)

also, $\xi_n = 2\xi_{n-1} - \lambda'(y)$

but (vi) implies that $\lambda(y) = \lambda'(y)$

so $\lambda(y) = \lambda'(y) = 2\xi_{n-1} - \xi_n$ (2)

(1) and (2) lead to

$$\lambda(e) = \frac{\xi_n + 2\xi_{n-1} - \xi_n}{2} = \xi_{n-1}.$$

How do we determine ξ_n? In order to answer the question, first
let us consider those expanded trees on $2n$ vertices whose core trees
are simply paths on n vertices, as in figure 36. Such trees have
exactly two remote end vertices.
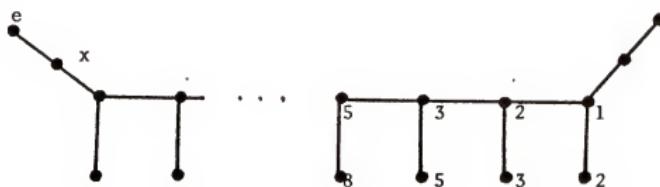
Figure 36

We will call the central n - 2 points of the core tree the <u>central path</u>, and will find the λ-numbers for all points of the central path, as well as for the non-remote end vertices.

Starting at the right hand end of the central path, we label each vertex of the central path and each corresponding end vertex with the number of maximal independent sets containing the point that include only points which have been previously labeled, or points only "to the right" of the given point. Points "to the right" of an end vertex shall include the point in the core tree to which it is connected. These labels will be elements of the Fibonacci sequence $(1,1,2,3,5,8,\ldots$ is the Fibonacci sequence, where the nth term, f_n, equals the sum of the two previous terms: $f_n = f_{n-1} + f_{n-2})$. Refer to the labels of figure 36.

In figure 37 is a portion of the tree in figure 36 with labels that will correspond to the following discussion.
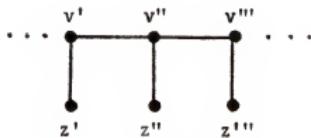
Figure 37

Define r(w) to be the number of MIS's containing vertex w and points "to the right" of w.

For a vertex v' in the central path, v' is in the same number of maximal independent sets to its right as z" is in, where z" is the endpoint adjacent to v", v" adj v' and v" to the right of v'. $r(z') = r(v'') + r(z'')$ since if z' is in a MIS M (containing points only "to the right" of v'), then either z" ε M or v" ε M.

Therefore, $r(v') = r(z'') = r(z''') + r(v''')$

$$= r(v'') + r(v''').$$

Thus if we number the vertices of the central path from left to right by v_{n-2}, v_{n-3}, ..., v_2, v_1, and the end vertices by z_{n-2}, z_{n-3}, ..., z_2, z_1, where z_i adj v_i, then $r(v_i) = f_{i+1}$, and $r(z_i) = f_{i+2}$. Then, $\lambda(v_{n-2}) = f_{n-1}$ and $\lambda(z_{n-2}) = 2 \cdot f_n$ (since for every MIS "to the right" we could add either the remote end vertex e or the adjacent point x) and hence,

$$\xi_n = \lambda(v_{n-2}) + \lambda(z_{n-2}) = f_{n-1} + 2f_n$$

$$= f_{n-1} + f_n + f_n = f_{n+1} + f_n = f_{n+2}.$$

Now label each point v_i and end point z_i in a similar manner from the left by $\ell(v_i)$ and $\ell(z_i)$. $\lambda(v_i) = r(v_i) \cdot \ell(v_i)$ and $\lambda(z_i) = r(z_i) \cdot \ell(z_i)$. Note that $\ell(v_i) = f_{n-i}$ since v_i is the $n - i - 1^{st}$ point from the left, and $\ell(z_i) = f_{n-i+1}$. Since $\lambda(v_i) + \lambda(z_i) = \xi_n$, $1 \leq i \leq n - 2$, we have the following well-known [6] number theoretic result:

Theorem 3.24: $f_{n+2} = f_{i+1} \; f_{n-i} + f_{i+2} \; f_{n-i+1}$ for $1 \leq i \leq n - 2$.

For more general expanded trees we follow a similar procedure. If v is a member of the core tree and deg $v = k$, then $\lambda(v)$ is the product of $k - 1$ labels -- one from each of the $k - 1$ branches incident with v. If v adj z, z an end vertex, then $\lambda(z)$ is also the product of $k - 1$ labels. In this general case, the labels will not always be elements of the Fibonacci sequence, but each individual label will be obtained as the sum of the two previous labels in the same branch. It is not necessary to find all labels for every point in order to find ξ_n. Only the λ-numbers for one end vertex and its adjacent point are needed.

In figure 38 is a tree with 20 vertices. The circled points are the ones for which the λ-numbers are being found. We are labeling from the endpoints of the separate branches <u>toward</u> the circled vertices.

$$\xi_{10} = 130 + 45 = 175$$
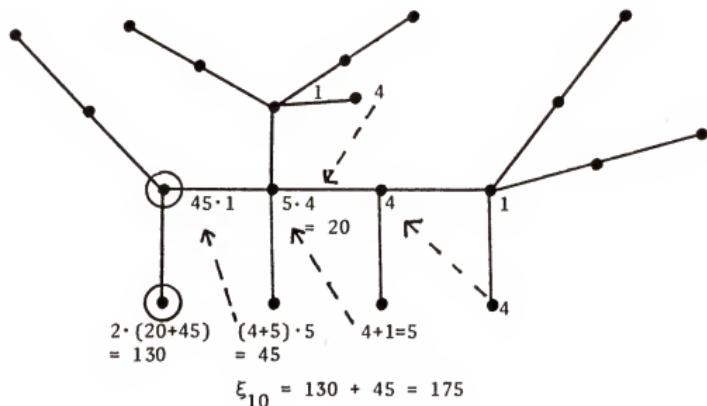
Figure 38

Other examples are in figures 39 and 40.

$$\xi_{10} = 114 + 35 = 149$$

Figure 39

$$\xi_{10} = 132 + 60 = 192$$
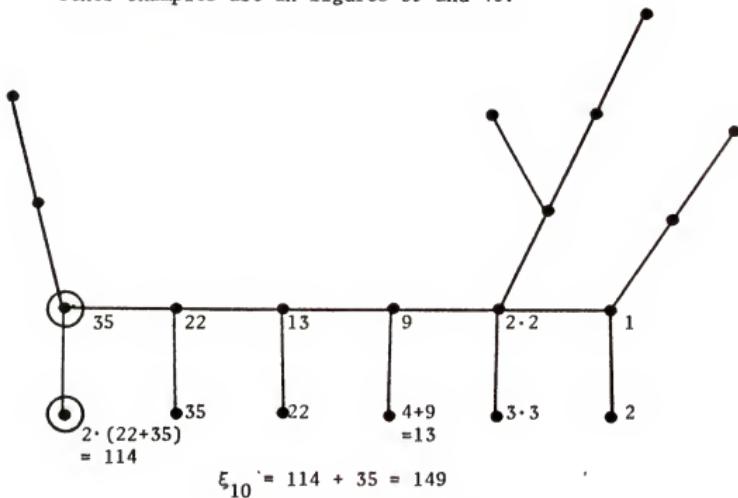
Figure 40
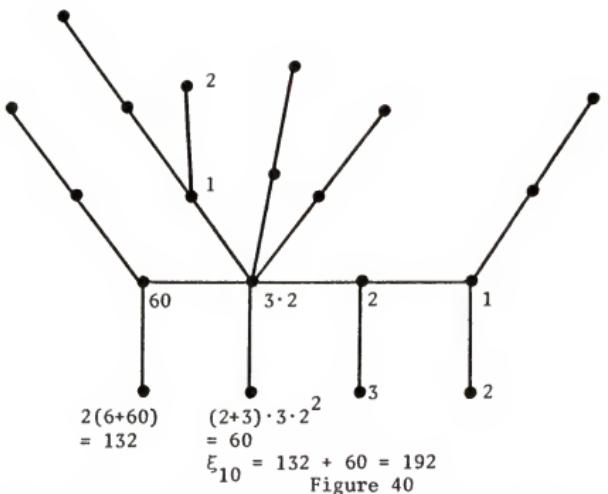
Observe that parts (i) and (ii) of Proposition 3.23 can be proved using the method of counting described in the discussion above.

Proposition 3.25: If e is a remote end vertex of an expanded tree T, and e adj x, x adj y, y ≠ e, then $\lambda(x) > \lambda(y)$.

Proof: Let z be the end vertex adjacent to y, and let y' adj y, y' ≠ z, y' ≠ x. Let f_v be the product of all labels at vertex v except the label for the process starting at e. Then

$$\lambda(x) = f_x = f_z = f_y + f_{y'} > f_y = \lambda(y)$$

Proposition 3.26: If T is an expanded tree whose core tree is a path on n vertices, and if e is a remote end vertex with x adj e, then $\lambda(x) > \lambda(v)$ for v a member of the central path.

Proof: $\lambda(x) = f_n = f_{i+1} f_{n-i} + f_i f_{n-i-1}$ by Theorem 3.24

$$\geq f_{i+1} f_{n-i} = \lambda(v_i), \quad 1 \leq i \leq n-1.$$

CHAPTER IV

THE NUMBER OF MAXIMAL INDEPENDENT SETS IN A TREE

Let $T = (V,E)$ be a tree with $|V| = n$ and with sequence $\sigma(T) = (\xi_1, \xi_2, \ldots, \xi_{n-1}; \beta)$ as defined in Chapter III. Dr. Paul Erdös raised the following question about $\sigma(T)$: What is $\displaystyle\max_{T:|V|=n} \sum_{i=1}^{n-1} \xi_i$?

Let $m_T = \displaystyle\sum_{i=1}^{n-1} \xi_i$ for T, $|V| = n$; and let $m(x) =$ number of maximal independent sets of T containing x, for $x \in V$.

Theorem 4.1: (i) If T is a forest with n vertices, then $\displaystyle\sum_{i=1}^{n-1} \xi_i = m_T = \leq 2^{\left[\frac{n}{2}\right]}$.

(ii) If T is a tree with n vertices and n is even, then
$$\max_{T:|V|=n} m_T = 2^{\frac{n-2}{2}} + 1.$$

Proof: The proof will proceed by induction.

(i) is true for $n = 1,2$. Assume the statement holds for all forests with fewer than n vertices and let T be a tree with n points. [Note that if T is a forest, its components $T_1, T_2, \ldots, T_k$ are trees and $m_T = m_{T_1} m_{T_2} \cdots m_{T_k}$ so we lose no generality by considering only trees.]

T has at least one end vertex a. There is some point b adjacent to a, and at least one point, c, adjacent to b.

$$m_T = m(a) + m(b).$$

But $m(a) =$ the number of maximal independent sets in $T - \{a,b\}$ since it is already determined for a set M containing a that $a \in M$ and $b \notin M$.

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Since T - {a,b} is a forest with n - 2 points the inductive hypothesis
implies that

$$m(a) = m_{T-\{a,b\}} \leq 2^{\lceil\frac{n-2}{2}\rceil}.$$

m(b) is less than or equal to the number of maximal independent sets
in T - {a,b,c} since if M contains b, then a $\notin$ M, b $\in$ M, c $\notin$ M are
determined. So

$$m(b) \leq m_{T-\{a,b,c\}} \leq 2^{\lceil\frac{n-3}{2}\rceil}.$$

Then

$$m_T \leq 2^{\lceil\frac{n-2}{2}\rceil} + 2^{\lceil\frac{n-3}{2}\rceil} \leq 2 \cdot 2^{\lceil\frac{n-2}{2}\rceil} = 2^{\lceil\frac{n}{2}\rceil}.$$

If T_1 is a forest on 2k vertices, $k \geq 1$, where each component
consists of two vertices and the edge between them, then $m_{T_1} = 2^k$. If
T_2 is a tree of the form in figure 41 with 2k + 1 vertices, then
$m_{T_2} = 2^k = 2^{\lceil\frac{2k+1}{2}\rceil}$, so the bound is the best possible.
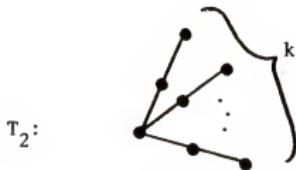
Figure 41

(ii) Let T be a tree with n vertices, n = 2k, $k \geq 1$ and assume
the statement is true for all trees with 2ℓ vertices, ℓ < k.

Case 1: T has a remote end vertex, e, and e is adjacent to x, degree
of x is 2, and x adj y, y ≠ e. T - {e,x} is a tree with n - 2 vertices
and $m_T = m(e) + m(x)$. As in part (i) $m(e) = m_{T-\{e,x\}}$ and $m(x) \leq$
$m_{T-\{e,x,y\}}$; and since n is even, so is n - 4, while n - 3 is odd.

Therefore

$$m_T \leq m_{T-\{e,x\}} + m_{T-\{e,x,y\}}$$
$$\leq (2^{[\frac{n-4}{2}]}+1) + 2^{[\frac{n-3}{2}]} \text{ by the inductive hypothesis and part (i)}$$
$$= 2^{\frac{n-4}{2}} + 1 + 2^{\frac{n-4}{2}} = 2 \cdot 2^{\frac{n-4}{2}} + 1 = 2^{\frac{n-2}{2}} + 1 = 2^{[\frac{n-2}{2}]} + 1.$$

<u>Case 2</u>: T has no remote end vertices. Then every end vertex is adjacent to a vertex of degree 3 or greater. If n > 5, T must look like the tree in figure 42. (Every tree has at least two endpoints, but since no endpoint is remote, and since the tree has only finitely many points, endpoints must come in pairs at least twice, as in the figure.)
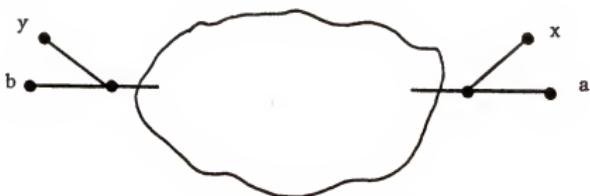
Figure 42

However $m_T = m_{T-\{x,y\}}$ since x and a must be in precisely the same maximal independent sets and the same is true for y and b. $T - \{x,y\}$ is a tree with $2(k-1)$ vertices so by the inductive hypothesis

$$m_T = m_{T-\{x,y\}} \leq 2^{\frac{2(k-1)-2}{2}} + 1 = 2^{k-2} + 1 < 2^{k-1} + 1 = 2^{\frac{n-2}{2}} + 1.$$

(note that if n = 1,2,3,4, the theorem can be observed to be true simply by inspecting all trees on 1, 2, 3, or 4 vertices.)

If T has the form of the tree in figure 43, then $m_T = 2^{k-1} + 1$, so the maximum is attained.

Figure 43

The lower bound for the total number of maximal independent sets in a tree is 2 and is attained in a star; that is, in a tree T, $|T| = n$, such that all but one point of T are endpoints, and the remaining point has degree $n - 1$.

The following examples are of trees on $n = 2k$ vertices with $m_T = 2^{k-1} + 1$ other than the one in figure 43. In all three examples, $|T| = 2k$. (These are not necessarily the only other trees with $m_T = 2^{k-1} + 1$ when $|T| = 2k$.)

Example 4.2:
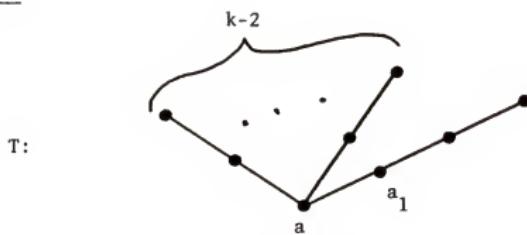
Figure 44

$$m_T = m(a) + m(\sim a) = m(a) + m(a_1) + m(\sim a_1)$$
$$= 2 + 2^{k-2} + (2^{k-2} - 1) \ (1)$$
$$= 2 \cdot 2^{k-2} + 1 = 2^{k-1} + 1.$$
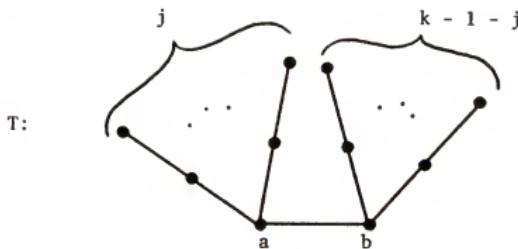
Example 4.3:

Figure 45

$$m_T = m(a,{\sim}b) + m({\sim}a,b) + m({\sim}a,{\sim}b)$$
$$= 2^{k-1-j} + 2^j + (2^j-1)(2^{k-1-j}-1)$$
$$= 2^{k-1-j} + 2^j + 2^{k-1} - 2^{k-1-j} - 2^j + 1$$
$$= 2^{k-1} + 1.$$

Example 4.4:
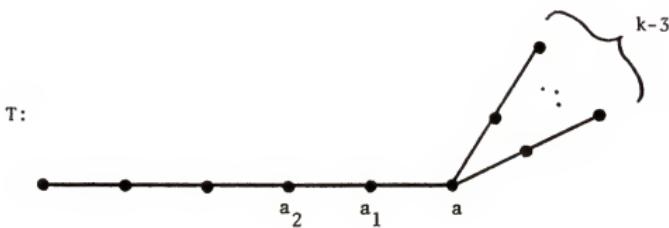
$$m_T = m(a) + m({\sim}a) = m(a) + m(a_1) + m(a_2)$$
$$= 1 \cdot 3 + (2^{k-3}-1) \cdot 2 + 2^{k-3} \cdot 2 = 2 \cdot 2 \cdot 2^{k-3} - 2 + 3$$
$$= 2^{k-1} + 1.$$

The result of Theorem 4.1 is a slight refinement of a similar result by Erdös, proved in a paper by Moon and Moser [5]. Recall that a complete graph C is a graph in which every vertex is adjacent to every other vertex. A complete graph C is said to be maximal with

respect to graph M if $C \subseteq M$ and C is not contained in any other complete graph contained in M. A <u>clique</u> is a complete graph C which is maximal with respect to G. Let $f(n)$ be the number of cliques possible in a graph with n nodes.

<u>Theorem 4.2</u>: If $n \geq 2$, then

$$f(n) = \begin{cases} 3^{\frac{n}{3}} & \text{if } n \equiv 0 \pmod 3 \\ 4 \cdot 3^{[\frac{n}{3}]-1} & \text{if } n \equiv 1 \pmod 3 \\ 2 \cdot 3^{[\frac{n}{3}]} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

If G is a graph on n vertices, and if G' is the complement of G with respect to K_n, then the points that form a clique in G form a maximal independent set in G' (and vice versa). Thus Theorem 4.2 gives a bound for the number of maximal independent sets of a graph and hence of a tree. However, the bounds of Theorem 4.1 are less than the bounds of Theorem 4.2. (This is easily proved by induction.) So what does Theorem 4.1 say about cliques? It says that the largest possible number of cliques in a graph G whose complement is acyclic (with n - 1 or fewer lines) is $2^{[\frac{n}{2}]}$.

CHAPTER V

DISCUSSION

Although much information about tree structure was gleaned from
the study of the sequence $\sigma(T) = (\xi_1, \xi_2, \ldots, \xi_{n-1}; \beta)$ for $|T| = n$, the
sequence, as shown in Chapter III, does not characterize trees. In
fact, the sequence when combined with several other facts about trees
still does not determine the structure. Therefore, two questions
that naturally arise are "Is there a sequence which will characterize
trees?" and "Is there a characterizing sequence which includes all
or part of $\sigma(T)$?" As mentioned previously, since there are computer
algorithms available to print out $\sigma(T)$, it would be especially convenient
if a characterizing sequence did contain the ξ_i, $i = 1, 2, \ldots, n-1$.

There is a theorem by Bednarek [1] that characterizes tree
isomorphisms:

Theorem 5.1: If T_1 and T_2 are trees and f: $T_1 \rightarrow T_2$ is a bijection
between vertex sets, then f is an isomorphism if and only if all
irreducible cycles in $T_1 - f - T_2$ have length four, where $T_1 - f - T_2$
is the graph consisting of T_1, T_2, and the edges of the bijection.
This theorem provides a possible way to prove (or disprove) that two
trees with the same sequence are isomorphic.

There are other questions arising from this study. In view of
the ease with which we can calculate $\sigma(p(T))$ for some expanded tree
$p(T)$ of T, can we use this plus the relationship between $\sigma(T)$ and

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$\sigma(p(T))$ to get information about T? In particular, can we use such a relationship to "transfer" information about a given sequence's ability to determine tree structure, as we did with $\sigma(T)$? Of course both of these questions presuppose that there _is_ a relationship between $\sigma(T)$ and $\sigma(p(T))$. As yet, although we can get a bound on $\sigma(p(T))$ from $\sigma(T)$, no equation has been found. A listing of $\sigma(T)$ and $\sigma(p(T))$ for $|T| = 2$, 3, 4, 5, 6, 7, 8, 9, 10 appears in Appendix III.

Another question was raised by Erdös. How many of the ξ_i in $\sigma(T)$ can be non-zero for a given cardinality? The question is analogous to the one about cliques in graphs, also discussed in [5] by Moon and Moser. If $g(n)$ is the maximum number of different sizes of cliques that could occur in a graph G on n nodes, then $g(n)$ is approximately $n - [\log_2 n]$. It follows then that approximately $n - [\log_2 n]$ is the maximum number of nonzero ξ_i. But can this be improved for trees? A more general, and much harder, problem is "What kind of sequences can arise as $\sigma(T)$?"

In this entire paper, the graphs have been finite, by definition. If we turn our attention to "infinite graphs," not all of the same questions can be asked. However, one that does arise is "If T is an infinite tree such that each finite subgraph of T with an even number of vertices is bivariegated, is T also bivariegated?" (Of course, the question may also be phrased in terms of 1-factors.)

APPENDIX I: COUNTEREXAMPLES

Here are some examples to show that if T_1 and T_2 are trees with $|T_1| = |T_2| = 2n$ and $\sigma(T_1) = \sigma(T_2) = (\xi_n; n)$, then it is not necessarily true that T_1 and T_2 are isomorphic.

Under each graph is the sequence $\sigma(T)$ for the <u>core tree</u> of the tree pictured; under each set of (expanded) trees having the same sequence is that sequence, denoted simply by σ. The values for σ were calculated using the method described in Chapter III.

While these examples are all such examples for expanded trees, there may be other (non-expanded) trees which are not isomorphic, but which have identical sequences.
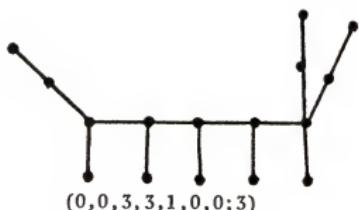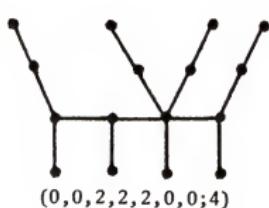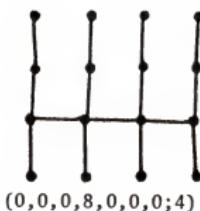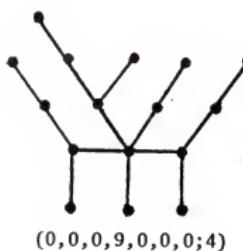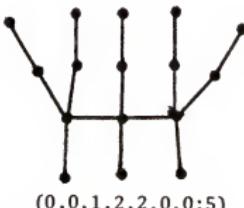
(0,0,3,3,1,0,0;3)

(0,0,0,8,0,0,0;4)

$\sigma = (60;8)$

(0,0,2,2,2,0,0;4)

(0,0,1,2,2,0,0;5)

$\sigma = (66;8)$

(0,0,0,9,0,0,0;4)

(0,0,1,4,1,0,0;4)

$\sigma = (62;8)$

(0,0,3,2,4,0,0,0;3)

(0,0,0,15,1,0,0,0;4)

$\sigma = (97;9)$

(0,0,3,0,3,1,0,0;4)   (0,0,0,3,2,1,0,0;5)

σ = (112;9)

(0,0,1,1,6,0,0,0;4)   (0,0,1,4,1,1,0,0;4)   (0,0,0,4,5,0,0.0;5)

σ = (106;9)

(0,0,0,6,4,0,0,0;4)   (0,0,1,2,5,0,0,0;4)

σ = (102;9)

(0,0,0,1,8,0,0,0;5)   (0,0,0,1,3,1,0,0;6)

σ = (116;9)

(0,0,1,4,6,1,0,0,0;3)

(0,0,0,11,4,1,0,0,0;4)

$\sigma = (157;10)$

(0,0,0,5,9,0,0,0,0;3)

(0,0,0,11,3,1,0,0,0;3)

$\sigma = (152;10)$

(0,0,1,2,10,0,0,0,0;3)

(0,0,0,9,8,0,0,0,0;4)

$\sigma = (153;10)$

(0,0,0,3,6,2,0,0,0;4)

(0,0,0,1,6,2,0,0,0;5)

$\sigma = (172;10)$

(0,0,2,2,3,3,0,0,0;4)   (0,0,2,1,3,3,0,0,0;5)

σ = (175;10)

(0,0,0,1,13,0,0,0,0,;4)   (0,0,0,7,3,2,0,0,0;4)

σ = (164;10)

(0,0,0,6,4,2,0,0,0;4)   (0,0,0,0,14,0,0,0,0;5)   (0,0,0,6,4,2,0,0,0;5)

σ = (168;10)

(0,0,3,0,0,3,1,0,0;5)   (0,0,0,0,2,2,1,0,0;7)
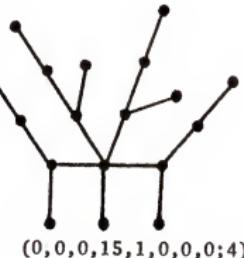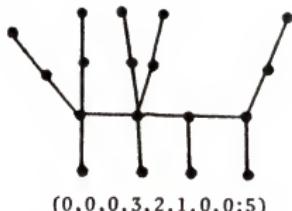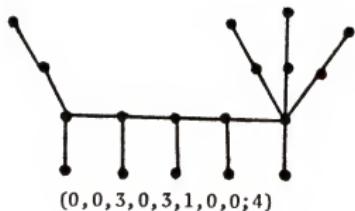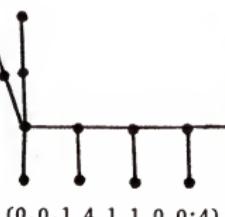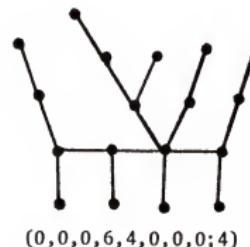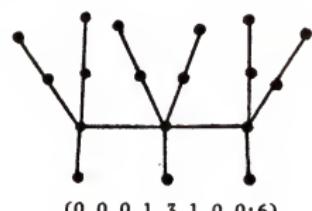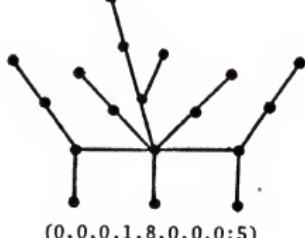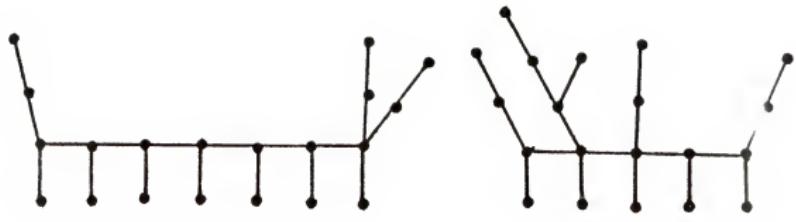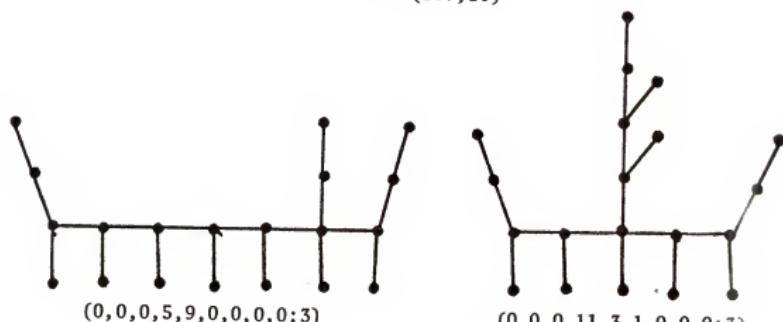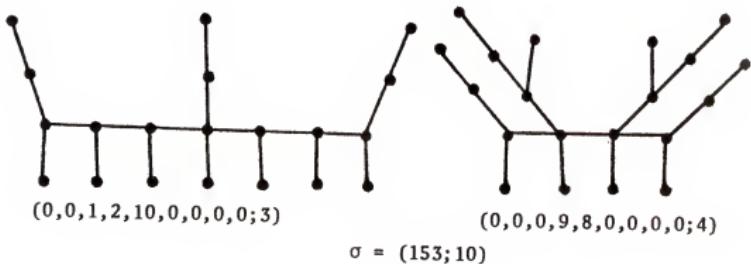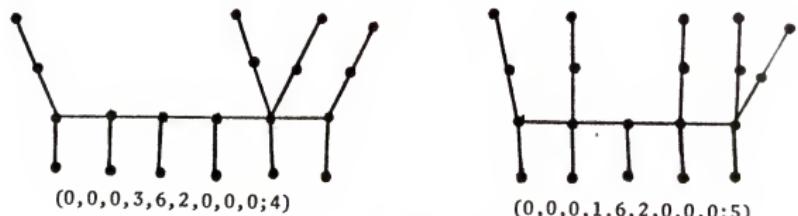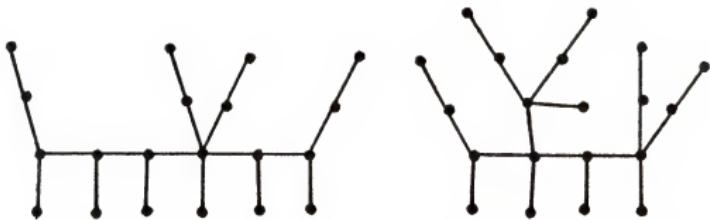
σ = (216;10)
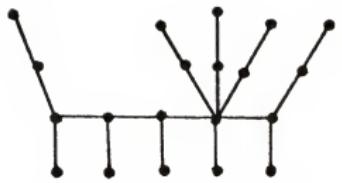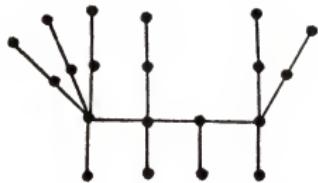
(0,0,1,1,0,6,0,0,0;5)   (0,0,1,0,3,1,1,0,0;6)

σ = (202;10)

(0,0,0,2,4,4,0,0,0;5)   (0,0,0,0,4,4,0,0,0;6)   (0,0,0,3,0,5,0,0,0;6)

σ' = (192; 10)

(0,0,0,0,17,0,0,0,0;5)   (0,0,0,4,3,3,0,0,0;5)

σ = (178;10)

(0,0,0,5,4,2,0,0,0;4)   (0,0,0,3,8,1,0,0,0;5)

σ = (166;10)

(0,0,0,5,7,1,0,0,0;4)   (0,0,0,3,7,1,0,0,0;4)

σ = (162;10)

(0,0,0,1,12,0,0,0,0;4)   (0,0,0,6,6,1,0,0,0;4)

(0,0,0,5,6,1,0,0,0;4)

σ = (160;10)

(0,0,0,3,11,0,0,0,0;4)   (0,0,0,2,11,0,0,0,0;4)

σ = (158;10)

$(0,0,3,1,2,1,1,0,0;5)$

$(0,0,0,8,7,1,0,0,0;5)$

$\sigma = (197;10)$

$(0,0,3,1,0,5,0,0,0;5)$

$(0,0,1,3,3,0,1,0,0;6)$

$\sigma = (189;10)$

$(0,0,0,3,2,4,0,0,0;5)$ $(0,0,0,0,5,3,0,0,0;6)$ $(0,0,0,3,1,4,0,0,0;5)$

$\sigma = (184;10)$

$(0,0,0,5,1,4,0,0,0;5)$

$(0,0,0,2,4,3,0,0,0;6)$
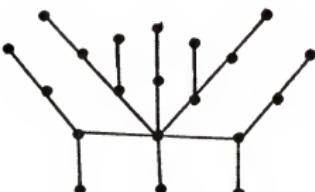
$\sigma = (182;10)$
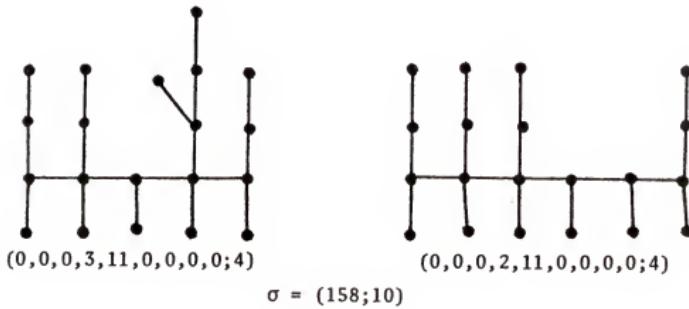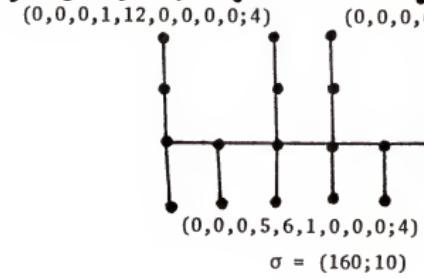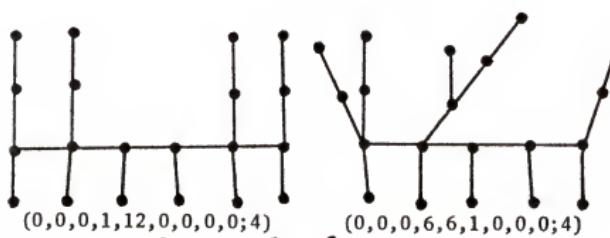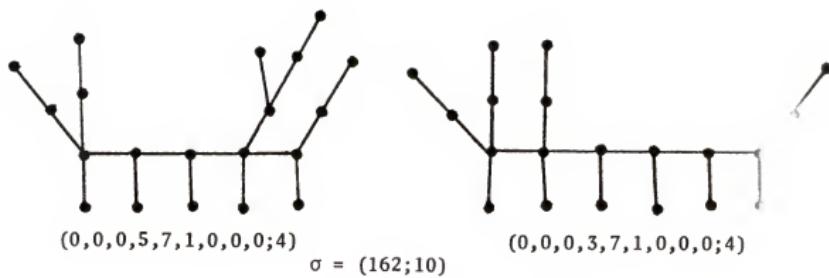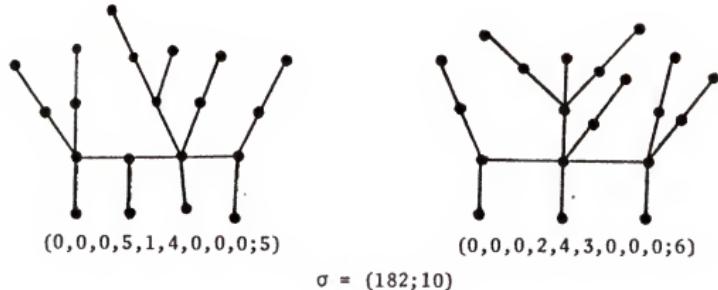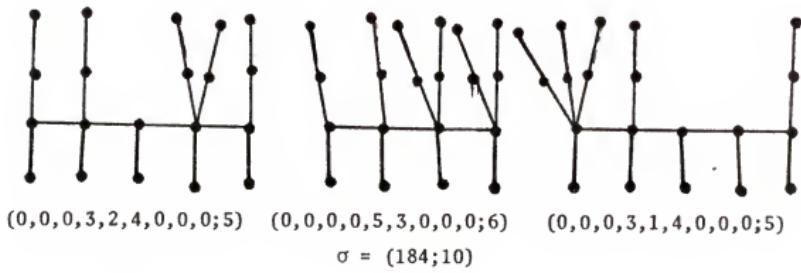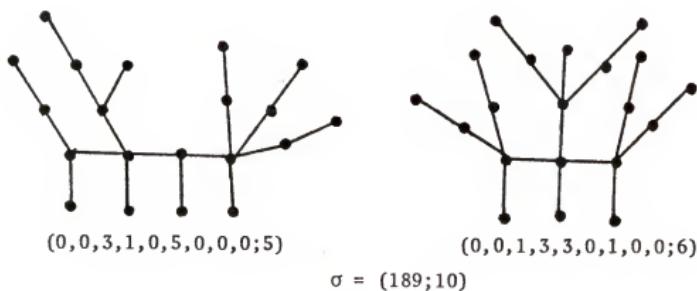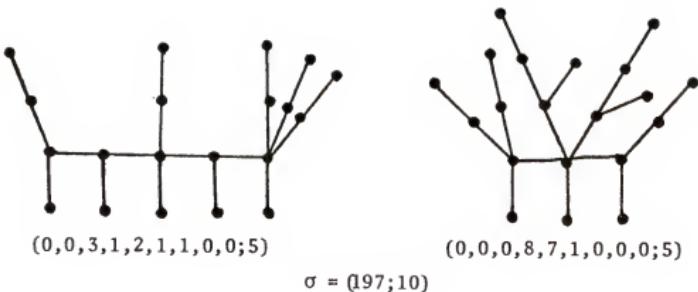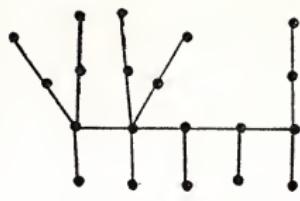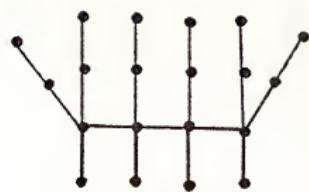
$(0,0,0,1,4,3,0,0,0;5)$ $(0,0,0,1,4,3,0,0,0;6)$

$\sigma = (180;10)$

$(0,0,2,2,1,4,0,0,0;5)$ $(0,0,1,3,2,3,0,0,0;5)$

$\sigma = (173;10)$
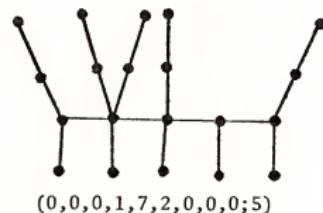
$(0,0,0,1,7,2,0,0,0;5)$ $(0,0,0,2,3,3,0,0,0;5)$

$\sigma = (176;10)$

$(0,0,3,0,0,1,2,0,0;6)$ $(0,0,0,0,0,3,0,2,0,0;7)$

$\sigma = (232;10)$

$(0,0,0,2,6,2,0,0,0;5)$ $(0,0,0,4,6,2,0,0,0;5)$ $(0,0,0,0,13,0,0,0,0;5)$

$(0,0,0,5,2,3,0,0,0;5)$

$\sigma = (174;10)$

$(0,0,1,1,2,0,2,0,0;6)$ $(0,0,1,0,2,0,2,0,0;7)$

$\sigma = (226;10)$

$(0,0,0,2,1,2,1,0,0;6)$ $(0,0,0,1,1,2,1,0,0;7)$

$\sigma = (212;10)$

$(0,0,0,2,0,6,0,0,0;6)$ $(0,0,0,1,3,1,1,0,0;6)$
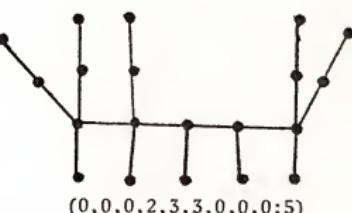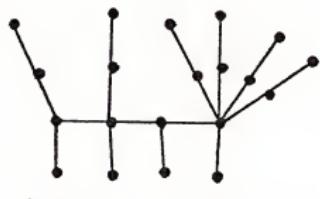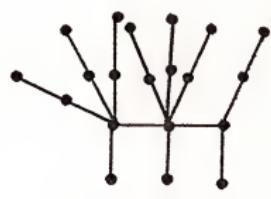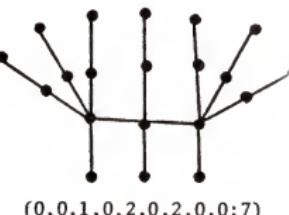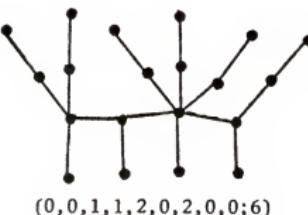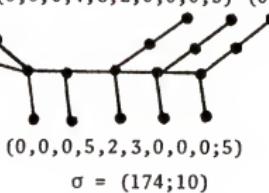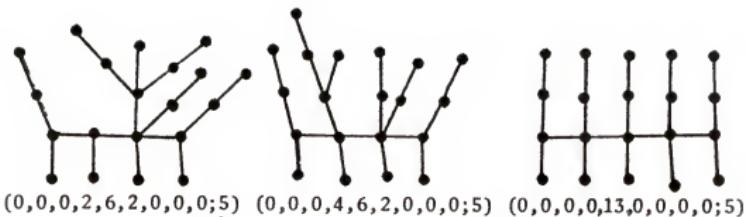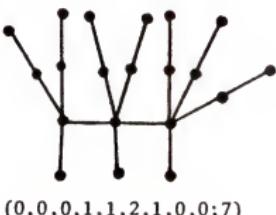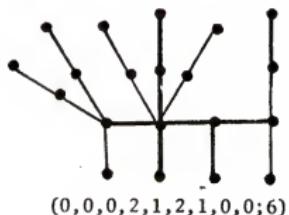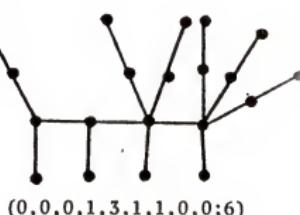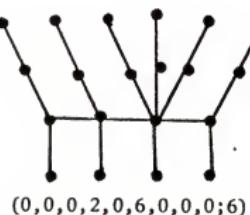
$\sigma = (204;10)$

APPENDIX II: BIVARIEGATED TREES

This appendix includes a table comparing the number of trees on
p points with the number of bivariegated trees on p points for p = 2, 4,
6, 8, 10, 12, and then shows all bivariegated trees on 12 or fewer
vertices.

TABLE 1

The number of trees (T_p) and bivariegated trees (B_p) with p points.

p	T_p	B_p
2	1	1
4	2	1
6	6	2
8	23	5
10	106	15
12	551	49

The listing of bivariegated trees, with up to 12 vertices was extracted from the listing in [4] of trees with at most 10 vertices and that in [8] of trees with 12 vertices. This listing also appeared in [2] and [9].

l_1 l_2 l_3 l_4 l_{10}

TABLE 2

n	$\sigma(T)$, $|T|$ = n	$\sigma(p(T)) = (\xi_n;n)$
2	(2;2)	(3;2)
3	(1,2;2)	(5;3)
4	(0,3,0;2)	(8;4)
	(1,0,1;3)	(9;4)
5	(0,3,1,0;2)	(13;5)
	(0,1,2,0;3)	(14;5)
	(1,0,0,1;4)	(17;5)
6	(0,1,4,0,0;2)	(21;6)
	(0,2,1,1,0;3)	(23;6)
	(0,0,5,0,0;3)	(22;6)
	(0,1,0,2,0;4)	(26;6)
	(1,0,0,0,1;5)	(33;6)
	(0,0,2,1,0;4)	(24;6)
7	(0,0,6,1,0,0;2)	(34;7)
	(0,1,1,3,0,0;3)	(37;7)
	(0,0,4,2,0,0;3)	(36;7)
	(0,0,2,3,0,0;4)	(38;7)
	(0,0,1,4,0,0;4)	(40;7)
	(0,0,7,1,0,0;3)	(35;7)
	(0,1,2,0,1,0;4)	(41;7)
	(0,2,0,1,1,0;4)	(43;7)
	(0,0,1,1,1,0;5)	(44;7)
	(0,1,0,0,2,0;5)	(50;7)
	(1,0,0,0,0,1;6)	(65;7)
8	(0,0,4,5,0,0,0;2)	(55;8)
	(0,0,3,3,1,0,0;3)	(60;8)
	(0,0,1,7,0,0,0;3)	(58;8)
	(0,0,3,6,0,0,0;3)	(57;8)
	(0,0,5,2,1,0,0;3)	(59;8)
	(0,1,1,0,3,0,0;4)	(69;8)
	(0,0,2,2,2,0,0;4)	(66;8)
	(0,0,0,9,0,0,0;4)	(62;8)
	(0,2,0,0,1,1,0;5)	(83;8)
	(0,0,1,0,4,0,0;5)	(76;8)
	(0,1,0,0,0,2,0;6)	(98;8)

$\underline{n}$	$\sigma(T),\ |T| = n$	$\sigma(p(T)) = (\xi_\eta; n)$
	(1,0,0,0,0,0,1;7)	(129;8)
	(0,1,0,2,2,0,0;4)	(65;8)
	(0,0,3,1,2,0,0;4)	(64;8)
	(0,0,4,3,1,0,0;4)	(61;8)
	(0,0,0,8,0,0,0;4)	(60;8)
	(0,0,1,4,1,0,0;4)	(62;8)
	(0,1,1,1,0,1,0;5)	(77;8)
	(0,0,0,3,2,0,0;5)	(68;8)
	(0,0,2,0,3,0,0;5)	(70;8)
	(0,0,1,0,1,1,0;6)	(82;8)
	(0,0,0,2,0,1,0;6)	(80;8)
	(0,0,1,2,2,0,0;5)	(66;8)
9	(0,0,1,10,1,0,0,0;2)	(89;9)
	(0,0,3,2,4,0,0,0;3)	(97;9)
	(0,0,0,9,2,0,0,0;3)	(94;9)
	(0,0,2,5,3,0,0,0;3)	(95;9)
	(0,0,0,12,1,0,0,0;3)	(92;9)
	(0,0,1,8,2,0,0,0;3)	(93;9)
	(0,0,3,0,3,1,0,0;4)	(112;9)
	(0,0,1,1,6,0,0,0;4)	(106;9)
	(0,0,4,1,2,1,0,0;4)	(109;9)
	(0,0,0,6,4,0,0,0;4)	(102;9)
	(0,0,0,15,1,0,0,0;4)	(97;9)
	(0,1,1,0,0,3,0,0;5)	(133;9)
	(0,0,2,0,2,2,0,0;5)	(126;9)
	(0,0,0,1,8,0,0,0;5)	(116;9)
	(0,2,0,0,0,1,1,0;6)	(163;9)
	(0,0,1,0,0,4,0,0;6)	(148;9)
	(0,1,0,0,0,0,2,0;7)	(194;9)
	(1,0,0,0,0,0,0,1;8)	(257;9)
	(0,0,1,4,1,1,0,0;4)	(106;9)
	(0,0,1,2,5,0,0,0;4)	(102;9)
	(0,0,3,1,5,0,0,0;4)	(101;9)
	(0,0,0,5,4,0,0,0;4)	(100;9)
	(0,0,3,3,1,1,0,0;4)	(105;9)
	(0,0,0,4,4,0,0,0;4)	(100;9)
	(0,0,2,3,4,0,0,0;4)	(99;9)
	(0,0,0,7,3,0,0,0;4)	(98;9)
	(0,0,0,10,2,0,0,0;4)	(96;9)
	(0,1,0,1,1,2,0,0;5)	(121;9)
	(0,0,1,3,0,2,0,0;5)	(118;9)
	(0,0,3,0,1,2,0,0;5)	(120;9)
	(0,0,0,3,2,1,0,0;5)	(112;9)
	(0,0,0,4,5,0,0,0;5)	(106;9)
	(0,1,0,2,0,0,1,0;6)	(145;9)
	(0,0,1,0,0,1,1,0;7)	(164;9)
	(0,0,2,0,0,3,0,0;6)	(134;9)
	(0,0,0,1,2,2,0,0;6)	(128;9)
	(0,1,1,0,1,0,1,0;6)	(149;9)
	(0,0,0,2,6,0,0,0;5)	(108;9)
	(0,0,4,0,3,1,0,0;5)	(113;9)

n	σ(T), |T| = n	σ(p(T)) = (ξ_n;n)
	(0,0,1,1,3,1,0,0;5)	(114;9)
	(0,0,0,2,1,2,0,0;6)	(124;9)
	(0,0,0,1,1,0,1,0;7)	(152;9)
	(0,0,0,3,5,0,0,0;5)	(104;9)
	(0,0,1,2,2,1,0,0;5)	(110;9)
	(0,0,2,4,1,1,0,0;5)	(107;9)
	(0,0,1,1,1,2,0,0;6)	(122;9)
	(0,0,0,1,3,1,0,0;6)	(116;9)
10	(0,0,0,10,6,0,0,0,0;2)	(144;10)
	(0,0,1,4,6,1,0,0,0;3)	(157;10)
	(0,0,0,5,9,0,0,0,0;3)	(152;10)
	(0,0,0,9,4,1,0,0,0;3)	(154;10)
	(0,0,0,10,7,0,0,0,0;3)	(149;10)
	(0,0,1,2,10,0,0,0,0;3)	(153;10)
	(0,0,0,7,8,0,0,0,0;3)	(150;10)
	(0,0,0,11,3,1,0,0,0;3)	(152;10)
	(0,0,3,1,1,4,0,0,0;4)	(181;10)
	(0,0,0,3,6,2,0,0,0;4)	(172;10)
	(0,0,2,2,3,3,0,0,0;4)	(175;10)
	(0,0,0,1,13,0,0,0,0;4)	(164;10)
	(0,0,0,6,4,2,0,0,0;4)	(168;10)
	(0,0,0,7,10,0,0,0,0;4)	(159;10)
	(0,0,3,0,0,3,1,0,0;5)	(216;10)
	(0,0,1,1,0,6,0,0,0;5)	(202;10)
	(0,0,4,0,1,2,1,0,0;5)	(215;10)
	(0,0,0,2,4,4,0,0,0;5)	(192;10)
	(0,0,0,0,17,0,0,0,0;5)	(178;10)
	(0,1,1,0,0,0,3,0,0;6)	(261;10)
	(0,0,2,0,0,2,2,0,0;6)	(246;10)
	(0,0,0,1,0,8,0,0,0;6)	(224;10)
	(0,2,0,0,0,0,1,1,0;7)	(323;10)
	(0,0,1,0,0,0,4,0,0;5)	(292;10)
	(0,1,0,0,0,0,0,2,0;8)	(386;10)
	(1,0,0,0,0,0,0,0,1;9)	(513;10)
	(0,0,2,2,2,3,0,0,0;4)	(191;10)
	(0,0,0,5,4,2,0,0,0;4)	(166;10)
	(0,0,0,5,7,1,0,0,0;4)	(162;10)
	(0,0,0,1,12,0,0,0,0;4)	(160;10)
	(0,0,2,1,5,2,0,0,0;4)	(167;10)
	(0,0,1,5,1,3,0,0,0;4)	(169;10)
	(0,0,1,4,4,2,0,0,0;4)	(165;10)
	(0,0,0,7,3,2,0,0,0;4)	(164;10)
	(0,0,0,3,11,0,0,0,0;4)	(158;10)
	(0,0,0,3,7,1,0,0,0;4)	(162;10)
	(0,0,0,6,6,1,0,0,0;4)	(160;10)
	(0,0,0,2,11,0,0,0,0;4)	(158;10)
	(0,0,1,2,7,1,0,0,0;4)	(161;10)
	(0,0,0,5,6,1,0,0,0;4)	(160;10)
	(0,0,0,4,10,0,0,0,0;4)	(156;10)
	(0,0,0,11,4,1,0,0,0;4)	(157;10)
	(0,0,0,9,8,0,0,0,0;4)	(153;10)
	(0,0,1,2,2,1,1,0,0;5)	(198;10)

$\underline{n}$	$\sigma(T), \; |T| = n$	$\sigma(p(T)) = (\xi_n; n)$
	$(0,0,1,2,0,5,0,0,0;5)$	$(190;10)$
	$(0,0,1,0,3,4,0,0,0;5)$	$(186;10)$
	$(0,0,3,1,2,1,1,0,0;5)$	$(197;10)$
	$(0,0,2,3,1,1,1,0,0;5)$	$(195;10)$
	$(0,0,3,1,0,5,0,0,0;5)$	$(189;10)$
	$(0,0,0,3,2,4,0,0,0;5)$	$(184;10)$
	$(0,0,0,5,1,4,0,0,0;5)$	$(182;10)$
	$(0,0,0,3,1,4,0,0,0;5)$	$(184;10)$
	$(0,0,0,1,4,3,0,0,0;5)$	$(180;10)$
	$(0,0,2,2,1,4,0,0,0;5)$	$(173;10)$
	$(0,0,0,1,7,2,0,0,0;5)$	$(176;10)$
	$(0,0,0,2,6,2,0,0,0;5)$	$(174;10)$
	$(0,0,0,4,6,2,0,0,0;5)$	$(174;10)$
	$(0,0,0,4,3,3,0,0,0;5)$	$(178;10)$
	$(0,0,0,0,14,0,0,0,0;5)$	$(168;10)$
	$(0,0,0,8,7,1,0,0,0;5)$	$(197;10)$
	$(0,1,0,1,0,1,2,0,0;6)$	$(233;10)$
	$(0,0,3,0,0,1,2,0,0;6)$	$(232;10)$
	$(0,0,1,1,2,0,2,0,0;6)$	$(226;10)$
	$(0,0,1,1,0,3,1,0,0;6)$	$(218;10)$
	$(0,0,0,2,1,2,1,0,0;6)$	$(212;10)$
	$(0,0,4,0,0,3,1,0,0;6)$	$(217;10)$
	$(0,0,0,2,0,6,0,0,0;6)$	$(204;10)$
	$(0,0,0,0,5,4,0,0,0;6)$	$(196;10)$
	$(0,1,1,0,0,1,0,1,0;7)$	$(293;10)$
	$(0,0,2,0,0,0,3,0,0;7)$	$(262;10)$
	$(0,0,0,1,0,2,2,0,0;7)$	$(248;10)$
	$(0,0,1,0,0,0,1,1,0;8)$	$(324;10)$
	$(0,1,0,0,2,0,2,0,0;6)$	$(225;10)$
	$(0,0,1,2,1,0,2,0,0;6)$	$(222;10)$
	$(0,0,0,4,0,5,0,0,0;6)$	$(194;10)$
	$(0,0,0,0,4,4,0,0,0;6)$	$(192;10)$
	$(0,0,0,1,3,1,1,0,0;6)$	$(204;10)$
	$(0,1,0,1,1,0,0,1,0;7)$	$(281;10)$
	$(0,0,0,2,0,1,2,0,0;7)$	$(236;10)$
	$(0,0,0,0,3,0,2,0,0;7)$	$(232;10)$
	$(0,0,0,1,0,1,0,1,0;8)$	$(296;10)$
	$(0,0,0,0,2,0,0,1,0;8)$	$(288;10)$
	$(0,0,1,3,2,3,0,0,0;5)$	$(173;10)$
	$(0,0,0,6,4,2,0,0,0;5)$	$(168;10)$
	$(0,0,0,3,5,2,0,0,0;5)$	$(170;10)$
	$(0,0,0,3,8,1,0,0,0;5)$	$(166;10)$
	$(0,0,0,0,13,0,0,0,0;5)$	$(174;10)$
	$(0,0,2,1,3,3,0,0,0;5)$	$(175;10)$
	$(0,0,0,1,6,2,0,0,0;5)$	$(172;10)$
	$(0,0,0,5,2,3,0,0,0;5)$	$(174;10)$
	$(0,0,0,2,3,3,0,0,0;5)$	$(176;10)$
	$(0,0,2,2,3,0,1,0,0;5)$	$(187;10)$
	$(0,0,0,2,4,3,0,0,0;6)$	$(182;10)$
	$(0,0,0,0,5,3,0,0,0;6)$	$(184;10)$
	$(0,0,0,2,2,1,1,0,0;6)$	$(200;10)$
	$(0,0,2,2,2,1,1,0,0;6)$	$(199;10)$
	$(0,0,0,1,3,4,0,0,0;6)$	$(188;10)$

n	$\sigma(T), \ |T| = n$	$\sigma(p(T)) = (\xi_n; n)$
	(0,0,0,3,0,5,0,0,0;6)	(192;10)
	(0,0,1,2,0,2,1,0,0;6)	(206;10)
	(0,0,1,0,3,1,1,0,0;6)	(202;10)
	(0,0,0,0,2,2,1,0,0;7)	(216;10)
	(0,0,1,0,2,0,2,0,0;7)	(226;10)
	(0,0,0,1,1,2,1,0,0;7)	(212;10)
	(0,0,1,3,3,0,1,0,0;6)	(189;10)
	(0,0,0,1,4,3,0,0,0;6)	(180;10)
	(0,0,1,1,0,1,2,0,0;7)	(234;10)

The sequences listed above are in the same order as, and correspond to, the trees which appear in the appendix of [4].

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BIOGRAPHICAL SKETCH

Esther Lee Knisley Sanders was born in Atlanta, Georgia, on December 20, 1949, and eleven days later moved to Johnson City, Tennessee, where she spent most of her pre-college life. In 1967, she graduated Validictorian from Science Hill High School in Johnson City. She attended Vanderbilt University as a National Merit Scholar, and majored in mathematics with a minor in psychology. In 1971, she graduated magna cum laude, and became a member of Phi Beta Kappa. Since then, she has been a graduate student in the Department of Mathematics at the University of Florida, working toward a doctorate after receiving a master's degree in 1973. She hopes to obtain a teaching job in Boston, and to continue research in Graph Theory.

She is married to Robert Alan Sanders, also from Johnson City, who will receive his Ph.D. in Chemistry in August 1975.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

J. Martinez
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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August, 1975

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